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Chapter-1

Getting Started

The ambiguous term **operations research (O.R.)** was coined during World War II, when the British military management called upon a group of scientists together to apply a scientific approach in the study of military operations to win the battle. The main objective was to allocate scarce resources in an effective manner to various military operations and to the activities within each operation. The effectiveness of operations research in military spread interest in it to other government departments and industry.

Due to the availability of faster and flexible computing facilities and the number of qualified O.R. professionals, it is now widely used in military, business, industry, transportation, public health, crime investigation, etc.

It is concerned with coordinating and controlling the operations or activities within an organization. O.R. can be regarded as use of mathematical and quantitative techniques to substantiate the decisions being taken. O.R takes tools from subjects like mathematics, statistics, engineering, economics, psychology, etc. and uses them to know the consequences of possible alternative actions.

Numerous synonyms for operations research are in common use. The British like **operational research** and the Americans like **management science**, but a preferable term to describe this subject is **decision analysis**.

DEFINITIONS:

O.R. is the art of winning wars without actually fighting. - Aurther Clarke

O.R. is concerned with scientifically deciding how to best design and operate man-machine systems usually under conditions requiring the allocation of scarce resources. - O.R. Society of America

O.R. is the art of giving bad answers to problems which otherwise have worse answers. - T.L. Saaty

O.R. is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problems. -D.W. Miller and M.K. Starr

O.R. is a scientific approach to problems solving for executive management. -H.M. Wagner

O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solution to the problem. -Churchman, Ackoff and Arnoff

O.R. is the study of administrative system pursued in the same scientific manner in which systems in Physics, Chemistry and Biology are studied in natural sciences.

O.R. is scientific methodology-analytical, experimental, quantitative-which by assessing the overall implication of various alternative courses of action in a management system, provides an improved basis for management decisions. – Pocock

O.R. is the application of the theories of Probability, Statistics, Queuing, Games, Linear Programming, etc. to the problems of war, govt. and industry.

O.R. is the use of scientific methods to provide criteria for decisions regarding man machine systems involving repetitive operations.

Phases and Processes of O.R.

• **Formulate the problem:** This is the most important process, it is generally lengthy and time consuming. The activities that constitute this step are visits, observations, research, etc. With the help of such activities, the O.R. scientist gets sufficient information and support to proceed and is better prepared to formulate the problem. This process starts with understanding of the organizational climate, its objectives and expectations. Further, the alternative courses of action are discovered in this step.

• **Develop a model:** Once a problem is formulated, the next step is to express the problem into a mathematical model that represents systems, processes or environment in the form of equations, relationships or formulas. We have to identify both the static and dynamic structural elements, and device mathematical formulas to represent the interrelationships among elements. The proposed model may be field tested and modified in order to work under stated environmental constraints. A model may also be modified if the management is not satisfied with the answer that it gives.

• **Select appropriate data input:** Garbage in and garbage out is a famous saying. No model will work appropriately if data input is not appropriate. The purpose of this step is to have sufficient input to operate and test the model.

• **Solution of the model:** After selecting the appropriate data input, the next step is to find a solution. If the model is not behaving properly, then updating and modification is considered at this stage.

• **Validation of the model:** A model is said to be valid if it can provide a reliable prediction of the system's performance. A model must be applicable for a longer time and can be updated from time to time taking into consideration the past, present and future aspects of the problem.

• **Implement the solution:** The implementation of the solution involves so many behavioral issues and the implementing authority is responsible for resolving these issues. The gap between one who provides a solution and one who wishes to use it should be eliminated. To achieve this, O.R. scientist as well as management should play a positive role. A properly implemented solution obtained through O.R. techniques results in improved working and wins the management support.

Techniques of O.R.

• **Linear Programming:** Linear Programming (LP) is a mathematical technique of assigning a fixed amount of resources to satisfy a number of demands in such a way that some objective is optimized and other defined conditions are also satisfied.

• **Transportation Problem:** The transportation problem is a special type of linear programming problem, where the objective is to minimize the cost of distributing a product from a number of sources to a number of destinations.

• **Assignment Problem:** Succinctly, when the problem involves the allocation of *n* different facilities to *n* different tasks, it is often termed as an assignment problem.

• **Queuing Theory:** The queuing problem is identified by the presence of a group of customers who arrive randomly to receive some service. This theory helps in calculating the expected number of people in the queue, expected waiting time in the queue, expected idle time for the server, etc. Thus, this theory can be applied in such situations where decisions have to be taken to minimize the extent and duration of the queue with minimum investment cost.

• **Game Theory:** It is used for decision making under conflicting situations where there are one or more opponents (i.e., players). In the game theory, we consider two or more persons with different objectives, each of whose actions influence the outcomes of the game. The game theory provides solutions to such games, assuming that each of the players wants to maximize his profits and minimize his losses.

• **Inventory Control Models:** It is concerned with the acquisition, storage, handling of inventories so as to ensure the availability of inventory whenever needed and minimize wastage and losses. It helps managers to decide reordering time, reordering level and optimal ordering quantity.

- **Goal Programming**: It is a powerful tool to tackle multiple and incompatible goals of an enterprise.
- **Simulation:** It is a technique that involves setting up a model of real situation and then performing experiments. Simulation is used where it is very risky, cumbersome, or time consuming to conduct real study or experiment to know more about a situation.
- **Nonlinear Programming:** These methods may be used when either the objective function or some of the constraints are not linear in nature. Non-Linearity may be introduced by factors such as discount on price of purchase of large quantities.
- **Integer Programming:** These methods may be used when one or more of the variables can take only integral values. Examples are the number of trucks in a fleet, the number of generators in a power house, etc.

- **Dynamic Programming:** Dynamic programming is a methodology useful for solving problems that involve taking decisions over several stages in a sequence. One thing common to all problems in this category is that current decisions influence both present & future periods.

- **Sequencing Theory:** It is related to Waiting Line Theory. It is applicable when the facilities are fixed, but the order of servicing may be controlled. The scheduling of service or sequencing of jobs is done to minimize the relevant costs. For example, patients waiting for a series of tests in a hospital, aircrafts waiting for landing clearances, etc.
- **Replacement Models**: These models are concerned with the problem of replacement of machines, individuals, capital assets, etc. due to their deteriorating efficiency, failure, or breakdown.
- **Markov Process:** This process is used in situations where various states are defined and the system moves from one state to another on a probability basis. The probability of going from one state to another is known. This theory helps in calculating long run probability of being in a particular state.
- **Network Scheduling-PERT and CPM:** Network scheduling is a technique used for planning, scheduling and monitoring large projects. Such large projects are very common in the field of construction, maintenance, computer system installation, research and development design, etc. Projects under network analysis are broken down into individual tasks, which are arranged in a logical sequence by deciding as to which activities should be performed simultaneously and which others sequentially.
- **Symbolic Logic:** It deals with substituting symbols for words, classes of things, or functional systems. It incorporates rules, algebra of logic, and propositions. There have been only limited attempts to apply this technique to business problems; however, it is extensively used in designing computing machinery.

• **Information Theory:** It is an analytical process transferred from the electrical communications field to operations research. It seeks to evaluate the effectiveness of information flow within a given system and helps in improving the communication flow.

Advantages & Limitations of O.R.

Advantages

- **Better Control:** The management of large organizations recognizes that it is a difficult and costly affair to provide continuous executive supervision to every routine work. An O.R. approach may provide the executive with an analytical and quantitative basis to identify the problem area. The most frequently adopted applications in this category deal with production scheduling and inventory replenishment.
- **Better Systems:** Often, an O.R. approach is initiated to analyze a particular problem of decision making such as best location for factories, whether to open a new warehouse, etc. It also helps in selecting economical means of transportation, jobs sequencing, production scheduling, replacement of old machinery, etc.
- **Better Decisions:** O.R. models help in improved decision making and reduce the risk of making erroneous decisions. O.R. approach gives the executive an improved insight into how he makes his decisions.

- **Better Co-ordination:** An operations-research-oriented planning model helps in coordinating different divisions of a company.

Limitations

- **Dependence on an Electronic Computer:** O.R. techniques try to find out an optimal solution taking into account all the factors. In the modern society, these factors are enormous and expressing them in quantity and establishing relationships among these require voluminous calculations that can only be handled by computers.
- **Non-Quantifiable Factors:** O.R. techniques provide a solution only when all the elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors that cannot be quantified find no place in O.R. models.
- **Distance between Manager and Operations Researcher:** O.R. being specialist's job requires a mathematician or a statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of O.R. Thus, there is a gap between the two.

- **Money and Time Costs:** When the basic data are subjected to frequent changes, incorporating them into the O.R. models is a costly affair. Moreover, a fairly good solution at present may be more desirable than a perfect O.R. solution available after sometime.
- **Implementation:** Implementation of decisions is a delicate task. It must take into account the complexities of human relations and behavior.

Some problems that can be analyzed by operations research approach are classified as follows:

1. Finance, Budgeting and Investments

- Credit policy analysis.
- Cash flow analysis.
- Dividend policies.
- Investment portfolios.

2. Marketing

- Product selection, timing, etc.
- Advertising media, budget allocation.
- Number of salesman required.
- Selection of product mix.

3. Purchasing, Procurement and Exploration

- Optimal buying and reordering.
- Replacement policies

4. Production Management

- Location and size of warehouses, factories, retail outlets, etc.
- Distribution policy.
- Loading and unloading facilities for trucks, etc.
- Production scheduling.
- Optimum product mix.
- Project scheduling and allocation of resources.

5. Personnel Management

- Selection of suitable personnel.
- Recruitment of employees.
- Assignment of jobs.

- Skills balancing.

6. Research and Development

- Project selection.
- Control of R&D projects.
- Reliability and alternative design.

Self Test Questions

1. What is Operations Research?

2. Describe the various steps involved in O.R. study.

3. Explain why it may be advantageous to build models to help in solving a decision problem.

4. Enumerate, with brief description, some of the techniques of O.R.

5. Discuss the advantages and limitations of O.R.

6. Discuss the significance and scope of O.R. in scientific management.

7. Identify some areas of application of O.R. technique in your organization.

Chapter-2

Model Formulation & Graphical Method

Introduction

In fact, every organization faces the problem of allocating limited resources to different activities. Such type of problem arises when there are alternative ways of performing a number of activities. For instance, consider a manufacturing firm where it is possible to manufacture a variety of products. Each of the products has a certain margin of profit per unit. These products use a common pool of resources whose availability is limited. Now the problem is to carefully allocate these resources to different types of finished products in such a way so that the total return

may be Maximum. In such a situation, the management's decision may be based on **past** experience and intuition, but decision so made is subjective rather than objective.

DEFINITION:

Linear Programming (LP) is a versatile technique for assigning a fixed amount of resources among competing factors, in such a way that some objective is optimized and other defined conditions are also satisfied. In other words, linear programming is a mathematical technique for determining the optimal allocation of resources and obtaining a particular objective when there are alternative uses of the resources. The objective may be cost minimization or inversely profit maximization.

Linear programming has been successfully applied to a variety of problems of management, such as production, advertising, transportation, refinery operation, investment analysis, etc. Over the years, linear programming has been found useful not only in business and industry but also in non-profit organizations such as government, hospitals, libraries, education, etc. Actually, linear programming improves the quality of decisions by amplifying the analytic abilities of a decision maker. Please note that the result of the mathematical models that you will study cannot substitute for the decision maker's **experience and intuition**, but they provide the comprehensive data needed to apply his knowledge effectively.

What is the meaning of linear programming?

The word 'linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form $k = p + qx$. In other words, it is used to describe the relationships among two or more variables, which are directly proportional. The word 'programming' is concerned with optimal allocation of limited resources.

Basic Terminology

The present section serves the purpose of building your vocabulary of the terms frequently employed in the description of Linear Programming Models.

Linear Function

A linear function contains terms each of which is composed of only a single, continuous variable raised to (and only to) the power of 1.

Objective Function

It is a linear function of the decision variables expressing the objective of the decision-maker. The most typical forms of objective functions are: maximize f(x) or minimize f(x).

Decision Variables

These are economic or physical quantities whose numerical values indicate the solution of the linear programming problem. These variables are under the control of the decision-maker and could have an impact on the solution to the problem under consideration. The relationships among these variables should be linear.

Constraints

These are linear equations arising out of practical limitations. The mathematical forms of the constraints are:
$f(x) \geq b$ or $f(x) \leq b$ or $f(x) = b$

Feasible Solution

Any non-negative solution which satisfies all the constraints is known as a feasible solution. The region comprising all feasible solutions is referred to as feasible region.

Optimal Solution

The solution where the objective function is maximized or minimized is known as optimal solution.

General Linear Programming Problem

Consider the following

Optimize (maximize or minimize)
$z = c_1x_1 + c_2x_2 + c_3x_3 + ...+ c_nx_n$

Subject to

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ...+ a_{1n}x_n \ (\leq, =, \geq) \ b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ...+ a_{2n}x_n \ (\leq, =, \geq) \ b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ...+ a_{mn}x_n \ (\leq, =, \geq) \ b_m$
$x_1, x_2,....., x_n \geq 0$

A word of guidance

If you have never taken a statistics course, then you will probably find the following $\sum$ notation strange, and perhaps even puzzling. To properly understand the text, read the text atleast twice.

In $\sum$ notation, it is written as

Optimize (maximize or minimize) $z = \sum_{j=1}^{n} c_jx_j$

Subject to

$\sum_{j=1}^{n} a_{ij}x_j \ (\leq, =, \geq) \ b_i; \ i = 1, 2, ..., m$ (constraints)

$x_j \geq 0; \ j = 1, 2, ..., n$ (non-negative restrictions)

Where all c_j's, a_{ij}'s, b_i's are constants and x_j's are decision variables. The expression (≤, =, ≥) means that each constraint may take only one of the three possible forms:

- less than or equal to (≤)
- equal to (=)
- greater than or equal to (≥)

The expression $x_j \geq 0$ means that the x_j's must be non-negative.

Model Formulation

In fact, the most difficult problem in the application of management science is the formulation of a model. Therefore, it is important to consider model formulation before launching into the details of linear programming solution. Model formulation is the process of transforming a real word decision problem into an operations research model. In the sections that follow, we give several Lilliputian examples so that you can acquire some experience of formulating a model. All the examples that we provide in the following sections are of static models, because they deal with decisions that occur only within a single time period.

Product Mix Problem

In product mix selection, the decision-maker strives to determine the combination of the products, which will maximize the profit without violating the resource constraints. In the instance below, the objective is to maximize the profit. In the subsequent examples, you will see other objectives, such as minimum cost.

Example -1

Universal Corporation manufactures two products- P_1 and P_2. The profit per unit of the two products is Rs. 50 and Rs. 60 respectively. Both the products require processing in three machines. The following table indicates the available machine hours per week and the time required on each machine for one unit of P_1 and P_2. Formulate this product mix problem in the linear programming form.

Machine	Product		Available Time (in machine hours per week)
	P_1	P_2	
1	2	1	300
2	3	4	509
3	4	7	812
Profit	Rs. 50	Rs. 60	

Solution

Let x_1 and x_2 be the amounts manufactured of products P_1 and P_2 respectively. The objective here is to **maximize** the profit, which is given by the linear function

Maximize $z = 50x_1 + 60x_2$

Since one unit of product P_1 requires two hours of processing in machine 1, while the corresponding requirement of P_2 is one hour, the first constraint can be expressed as
$2x_1 + x_2 \leq 300$

Similarly, constraints corresponding to machine 2 and machine 3 are
$3x_1 + 4x_2 \leq 509$
$4x_1 + 7x_2 \leq 812$

In addition, there **cannot** be any **negative production** that may be stated algebraically as
$x_1 \geq 0,\ x_2 \geq 0$

The problem can now be stated in the standard linear programming form as

Maximize $z = 50x_1 + 60x_2$

subject to

$2x_1 + x_2 \leq 300$
$3x_1 + 4x_2 \leq 509$
$4x_1 + 7x_2 \leq 812$

$x_1 \geq 0,\ x_2 \geq 0$

This procedure is commonly referred to as the formulation of the problem.

Assumptions

• **Additivity:** The total amounts of each input and the corresponding profit are the sums of the inputs and profit for each individual process. For example, one unit of Product 1 and one unit of Product 2 require 3 (2 + 1) hours of processing in machine 1, 7 (3 + 4) hours of processing in machine 2, 11 (4 + 7) hours of processing in machine 3, and yield Rs.110 (50 + 60) profit.
• **Divisibility:** The activity levels are allowed to assume fractional values as well as integer values. For example, we admit the possibility of x1 = 509/3 or 169.67, x2 = 509/4 or 127.25.

• **Proportionality:** The total amounts of each input and the associated profit are directly proportional to the level of output.

For example, to manufacture 1 unit of product 1 (x1 = 1), we require 2 hours of processing in machine 1, 3 hours of processing in machine 2, 4 hours of processing in machine 3, and the profit is Rs.50. By the same token, to manufacture 10 units of product 1 (x1 = 10), we require 20 hours of processing in machine 1, 30 hours of processing in machine 2, 40 hours of processing in machine 3, and the profit is Rs.500.

For all the examples in the subsequent sections of this chapter, we assume these assumptions to hold good.

Packaging Problem

Example -2

The Best Stuffing Company manufactures two types of packing tins- round & flat. Major production facilities involved are cutting and joining. The cutting department can process 200 round tins or 400 flat tins per hour. The joining department can process 400 round tins or 200 flat tins per hour. If the contribution towards profit for a round tin is the same as that of a flat tin, what is the optimal production level?

Solution.

Let
x_1 = number of round tins per hour
x_2 = number of flat tins per hour

Since the contribution towards profit is identical for both the products, the objective function can be expressed as $\mathbf{x_1 + x_2}$**.** Hence, the problem can be formulated as

Maximize $\mathbf{Z = x_1 + x_2}$

Subject to

$(1/200)x_1 + (1/400)x_2 \le 1$
$(1/400)x_1 + (1/200)x_2 \le 1$

$x_1 \ge 0, x_2 \ge 0$

i.e., $2x_1 + x_2 \le 400$
$x_1 + 2x_2 \le 400$

$x_1 \ge 0, x_2 \ge 0$

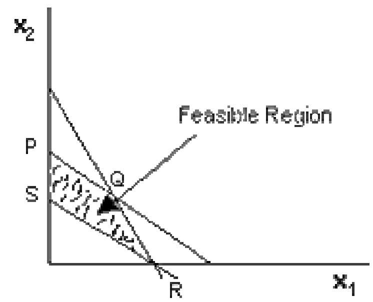
Graphical Method – Introduction

Linear programming problems with two decision variables can be easily solved by graphical method. A problem of three dimensions can also be solved by this method, but their graphical solution becomes complicated.

Basic Terminology

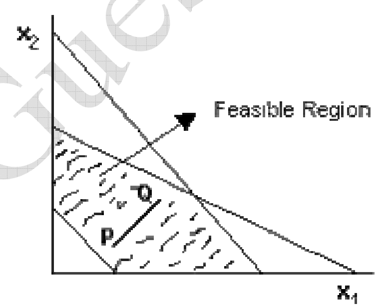
Feasible Region

It is the collection of all feasible solutions. In the following figure, the shaded area represents the feasible region.



Convex Set

A region or a set R is convex, if for any two points on the set R, the segment connecting those points lies entirely in R. In other words, it is a collection of points such that for any two points on the set, the line joining the points belongs to the set. In the following figure, the line joining P and Q belongs entirely in R.



Thus, the collection of feasible solutions in a linear programming problem form a convex set.

Redundant Constraint

It is a constraint that does not affect the feasible region.

Example: Consider the linear programming problem:

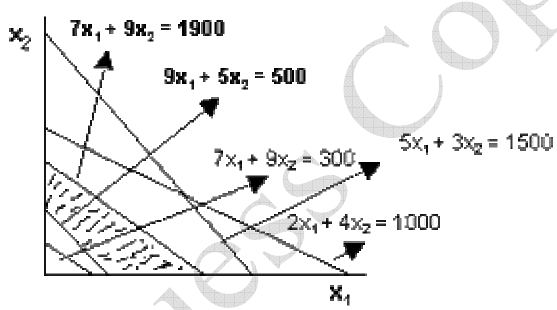
Maximize 1170 x_1 + 1110x_2

subject to

$9x_1 + 5x_2 \geq 500$
$7x_1 + 9x_2 \geq 300$
$5x_1 + 3x_2 \leq 1500$
$7x_1 + 9x_2 \leq 1900$
$2x_1 + 4x_2 \leq 1000$

$x_1, x_2 \geq 0$

The feasible region is indicated in the following figure:



The critical region has been formed by the two constraints.

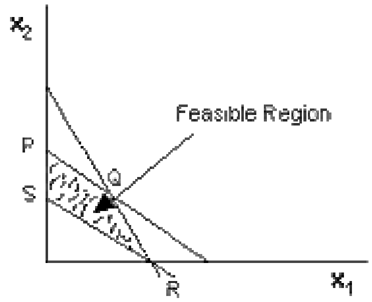
$9x_1 + 5x_2 \geq 500$
$7x_1 + 9x_2 \leq 1900$

$x_1, x_2 \geq 0$

The remaining three constraints are not affecting the feasible region in any manner. Such constraints are called **redundant constraints**.

Extreme Point

Extreme points are referred to as vertices or corner points. In the following figure, P, Q, R and S are extreme points.



Graphical Method – Algorithm

The details of the algorithm for this method are as follows.

1. Formulate the mathematical model of the given linear programming problem.
2. Treat inequalities as equalities and then draw the lines corresponding to each equation and non-negativity restrictions.
3. Locate the end points (corner points) on the feasible region.
4. Determine the value of the objective function corresponding to the end points determined in step 3.
5. Find out the optimal value of the objective function.

The examples to follow illustrate the method.

Graphical Method – Examples

Example 1

Maximize z = $18x_1 + 16x_2$

Subject to

$15x_1 + 25x_2 \leq 375$
$24x_1 + 11x_2 \leq 264$

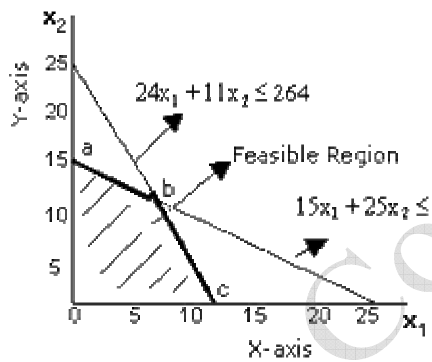
$x_1, x_2 \geq 0$

Solution

If only x_1 and no x_2 is produced, the maximum value of x_1 is 375/15 = 25. If only x_2 and no x_1 is produced, the maximum value of x_2 is 375/25 = 15. A line drawn between these two points (25, 0) & (0, 15), represents the constraint factor $15x_1 + 25x_2 \leq 375$. Any point which lies on or below this line will satisfy this inequality and the solution will be somewhere in the region bounded by it.

Similarly, the line for the second constraint $24x_1 + 11x_2 \le 264$ can be drawn. The polygon *oabc* represents the region of values for x_1 & x_2 that satisfy all the constraints. This polygon is called the solution set.

The solution to this simple problem is exhibited graphically below.



The end points (corner points) of the shaded area are (0,0), (11,0), (5.7, 11.58) and (0,15). The values of the objective function at these points are 0, 198, 288 (approx.) and 240. Out of these four values, 288 is maximum.

The optimal solution is at the extreme point b, where $x_1 = 5.7$ & $x_2 = 11.58$, and z = 288.

Example 2

Maximize $z = 6x_1 - 2x_2$

Subject to

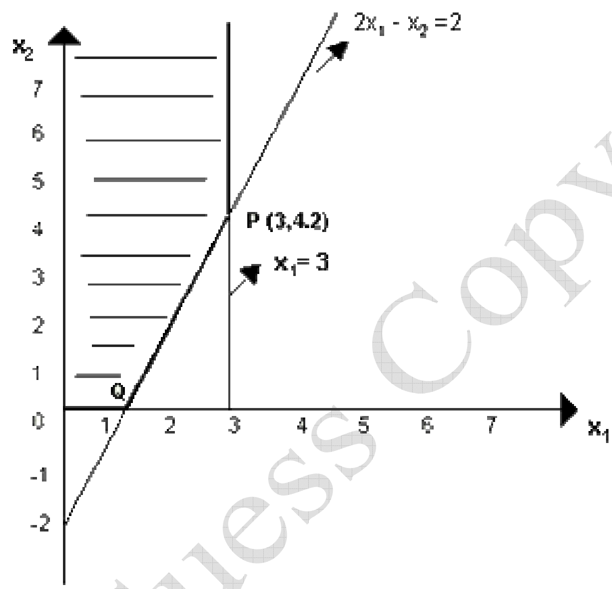
$2x_1 - x_2 \le 2$
$x_1 \le 3$

$x_1, x_2 \ge 0$

Solution

First, we draw the line $2x_1 - x_2 \leq 2$, which passes through the points (1, 0) & (0, -2). Any point which lies on or below this line will satisfy this inequality and the solution will be somewhere in the region bounded by it.

Similarly, the line for the second constraint $x_1 \leq 3$ is drawn. Thus, the optimal solution lies at one of the corner points of the dark shaded portion bounded by these straight lines.



Optimal solution is $x_1 = 3$, $x_2 = 4.2$, and the maximum value of z is 9.6.

Special Cases

1. Multiple Optimal Solutions

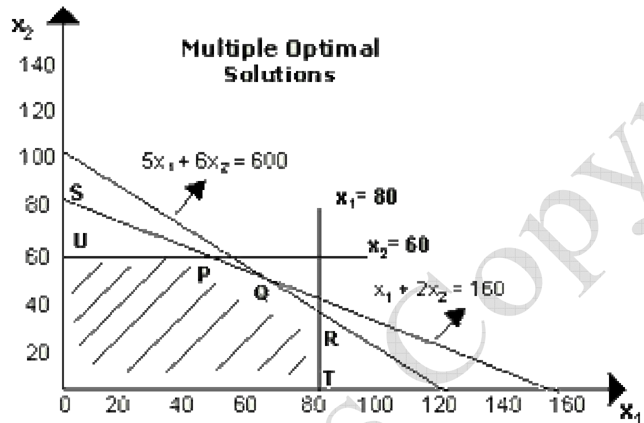
The linear programming problems discussed in the previous section possessed unique solutions. This was because the optimal value occurred at one of the extreme points (corner points). But situations may arise, when the optimal solution obtained is not unique. This case may arise when the line representing the objective function is parallel to one of the lines bounding the feasible region. The presence of multiple solutions is illustrated through the following example..

Maximize $z = x_1 + 2x_2$

subject to

$x_1 \le 80$
$x_2 \le 60$
$5x_1 + 6x_2 \le 600$
$x_1 + 2x_2 \le 160$

$x_1, x_2 \ge 0.$



In the above figure, there is no unique outer most corner cut by the objective function line. All points from P to Q lying on line PQ represent optimal solutions and all these will give the same optimal value (maximum profit) of Rs. 160. This is indicated by the fact that both the points P with co-ordinates (40, 60) and Q with co-ordinates (60, 50) are on the line $x_1 + 2x_2 = 160$. Thus, every point on the line PQ maximizes the value of the objective function and the problem has multiple solutions.

2. Infeasible Problem

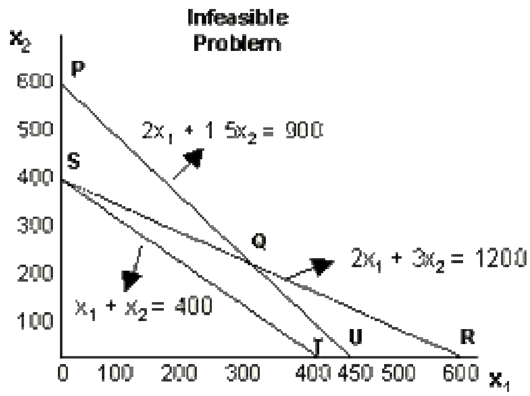
In some cases, there is no feasible solution area, i.e., there are no points that satisfy all constraints of the problem. An infeasible LP problem with two decision variables can be identified through its graph. For example, let us consider the following linear programming problem.

Minimize $z = 200x_1 + 300x_2$

Subject to

$2x_1 + 3x_2 \ge 1200$
$x_1 + x_2 \le 400$
$2x_1 + 1.5x_2 \ge 900$

$x_1, x_2 \geq 0$



The region located on the right of *PQR* includes all solutions, which satisfy the first and the third constraints. The region located on the left of *ST* includes all solutions, which satisfy the second constraint. Thus, the problem is infeasible because there is no set of points that satisfy all the three constraints.

3. Unbounded Solutions

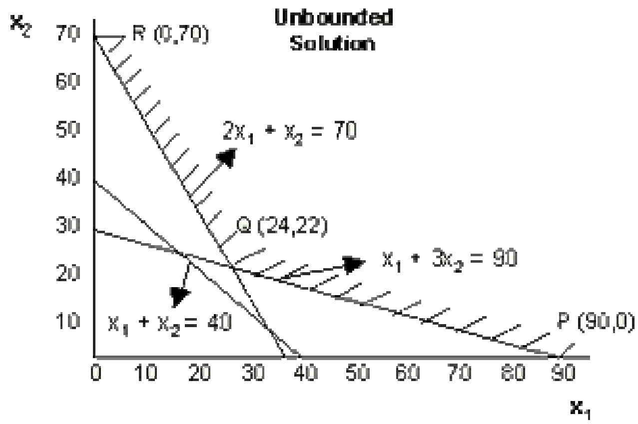
It is a solution whose objective function is infinite. If the feasible region is unbounded then one or more decision variables will increase indefinitely without violating feasibility, and the value of the objective function can be made arbitrarily large. Consider the following model:

Minimize $z = 40x_1 + 60x_2$

Subject to

$2x_1 + x_2 \geq 70$
$x_1 + x_2 \geq 40$
$x_1 + 3x_2 \geq 90$

$x_1, x_2 \geq 0$



The point (x_1, x_2) must be somewhere in the solution space as shown in the figure by shaded portion.

The three extreme points (corner points) in the finite plane are:
P = (90, 0); Q = (24, 22) and R = (0, 70)
The values of the objective function at these extreme points are:
Z(P) = 3600, Z(Q) = 2280 and Z(R) = 4200

In this case, no maximum of the objective function exists because the region has no boundary for increasing values of x_1 and x_2. Thus, it is not possible to maximize the objective function in this case and the solution is unbounded.

Limitations of Linear Programming

• **Linearity of relations:** A primary requirement of linear programming is that the objective function and every constraint must be linear. However, in real life situations, several business and industrial problems are nonlinear in nature.
• **Single objective:** Linear programming takes into account a single objective only, i.e., profit maximization or cost minimization. However, in today's dynamic business environment, there is no single universal objective for all organizations.
• **Certainty:** Linear Programming assumes that the values of co-efficient of decision variables are known with certainty. Due to this restrictive assumption, linear programming cannot be applied to a wide variety of problems where values of the coefficients are probabilistic.
• **Constant parameters:** Parameters appearing in LP are assumed to be constant, but in practical situations it is not so.

• **Divisibility:** In linear programming, the decision variables are allowed to take non-negative integer as well as fractional values. However, we quite often face situations where the planning models contain integer valued variables. For instance, trucks in a fleet, generators in a powerhouse, pieces of equipment, investment alternatives and there are a myriad of other examples. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases, linear programming techniques cannot be used.

Self Test Questions

Theory

1. Explain with the help of a suitable example, what do you understand by linear programming.

2. What are the characteristics and limitations of a linear programming problem?

3. What do you understand by graphical method? Give its limitations.

4. Fill in the blanks

i. A linear programming problem has a well defined objective function which is ...... and which is to be ...... or ....
ii. The constraints in a linear programming problem arises due to limitation of ...... These are linear ...... or ......
iii. The solution of a linear programming problem indicates the right combination of ...... which ...... or ...... the objective function satisfying the various ....

Practical

Problem Formulation

1. Jumpin Ltd. has canned apple and bottled juice as its products with profit margin Rs. 2 and Rs. 1 respectively per unit. The following table indicates the labour, equipment and material to produce each product per unit.

	Bottled Juice	**Canned Apple**	**Total**
Labour (man hours)	3	2	12
Equipment (machine hours)	1	2.3	6.9
Material (unit)	1	1.4	4.9

Formulate the problem specifying the product mix, which will maximize profit without exceeding the various levels of resources.

2. The managing director of a small scale company decides to manufacture two products, P1 & P2, each of which is processed in two shops, viz. Machining shop (M) and Finishing shop (F). One unit of P1 takes 15 hours of machining and 24 hours of finishing shops. The corresponding requirements for P2 are 25 hours and 11 hours respectively in shops. The total available hours per day in M and F shops are 375 and 264 respectively. P1 gives a profit of Rs. 18 per unit and P2 Rs. 16 per unit. Formulate the above problem.

3. A company owns two flour mills, A and B, which have different production capacities for high, medium and flour. This company has entered a contract to supply flour to a firm every week with at least 12, 8 and 24 quintals of high, medium and low grade respectively. It costs the company Rs. 1000 and Rs. 800 per day to run mill A and B respectively. On a day, mill A produce 6, 2 and 4 quintals of high, medium and low grade flour respectively. Mill B produce 2, 2 and 12 quintals of high, medium and low grade flour respectively. How many days per week should each mill be operated in order to meet the contract order most economically.

4. Decibel Electronics produces two products A and B that are sold on a weekly basis. The weekly production cannot exceed 25 for product A and 35 for product B. The company employs a total of 80 workers. Product A requires 2 man-weeks of labour whereas B requires only 1. A gives a profit of Rs. 16 and B Rs. 40. Formulate the above LPP.

5. A company that produces soft drinks has a contract that requires that a minimum of 80 units of the chemical A and 60 units of the chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X_1 has a mix of 4 units of A and 2 units of B that costs Rs. 10, and supplier X_2 has a mix of 1 units of A and 1 unit of B that costs Rs. 4. How many mixes from company X_1 and company X_1 should the company purchase to honour contract requirement and yet minimize cost?

Graphical Method

1. Maximize: $z = 3x + 2y$

subject to the constraints:

$x - y \leq 1$
$x + y \geq 3$

$x, y \geq 0$

2. Maximize: $z = x + y$

Subject to the constraints:

$x + y \leq 1$
$-3x + y \geq 3$

$x, y \geq 0$

3. A manufacture of packing material, manufacturers two type of packing tins, round and flat. Major production facilities involved are cutting and joining. The cutting department can process 300 round tins or 500 flat tins per hour. The joining department can process 500 round tins or 300 flat tins per hour. If the contribution towards profit for around tin is the same as that of a flat tin what is that the optimum production level?

4. Maximize $z = 3x_1 - 4x_2$

Subject to

$x_1 - x_2 \geq 0$
$x_2 \leq 6$

$x_1, x_2 \geq 0$

5. Maximize $z = 3x_1 + 2x_2$

subject to

$x_1 + 2x_2 \leq 2$
$2x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

6. Maximize $z = 4x_1 + 3x_2$

Subject to

$2x_1 + 3x_2 \leq 6$
$4x_1 + 6x_2 \geq 24$

$x_1, x_2 \geq 0$

7. Maximize $z = 3x_1 + 2x_2$

Subject to

$2x_1 - 3x_2 \geq 0$
$3x_1 + 4x_2 \leq -12$

$x_1, x_2 \geq 0$

8. Maximize $z = 50x_1 + 60x_2$

Subject to

$2x_1 + x_2 \leq 300$
$3x_1 + 4x_2 \leq 509$
$4x_1 + 7x_2 \leq 812$

$x_1, x_2 \geq 0$

9. Minimize $z = 2x_1 + 1.7x_2$

Subject to

$0.15x_1 + 0.10x_2 \geq 1.0$
$0.75x_1 + 1.70x_2 \geq 7.5$
$1.30x_1 + 1.10x_2 \geq 10.0$

$x_1, x_2 \geq 0$

10. Maximize $z = 3x_1 + 2x_2$

Subject to

$2x_1 - x_2 \geq 2$
$x_1 + 2x_2 \leq 8$

$x_1, x_2 \geq 0$

11. Minimize $z = x + y$

Subject to

$2x + y \geq 12$
$5x + 8y \geq 74$
$x + 6y \geq 24$

$x, y \geq 0$

12. Minimize $z = 2x_1 + 3x_2$

Subject to

$-x_1 + 2x_2 \leq 4$
$x_1 + x_2 \leq 6$
$x_1 + 3x_2 \geq 9$

$x_1, x_2 \geq 0$

13. Minimize $z = 3x + 4y$

Subject to

$5x + 8y \leq 2000$
$3x + 10y \leq 1000$

$x, y \geq 0$

14. Minimize $z = 7x_1 + 3x_2$

Subject to

$x_1 + 2x_2 \geq 3$
$x_1 + x_2 \leq 4$
$0 \leq x_1 \leq 15/2$
$0 \leq x_2 \leq 3/2$

15. Minimize $z = 4x_1 + 3x_2$

Subject to

$2x_1 + 3x_2 \leq 6$
$2x_2 \leq 5$
$-3x_1 + 2x_2 \leq 3$
$2x_1 + x_2 \leq 4 \qquad x_1, x_2 \geq 0$

Chapter – 3

Simplex Method

Introduction

In the previous chapter, we discussed about the graphical method for solving linear programming problems. Although the graphical method is an invaluable aid to understand the properties of linear programming models, it provides very little help in handling practical problems. In this chapter, we concentrate on the simplex method for solving linear programming problems with a larger number of variables.

Many different methods have been proposed to solve linear programming problems, but simplex method has proved to be the most effective. This technique will nurture your insight needed for a sound understanding of several approaches to other programming models, which will be studied in subsequent chapters. This method is applicable to any problem that can be formulated in terms of linear objective function, subject to a set of linear constraints. Often, this method is termed Dantzig's simplex method, in honour of the mathematician who devised the approach.

In the following section, we introduce you to the standard vocabulary of the simplex method.

Basic Terminology

Slack variable

It is a variable that is added to the left-hand side of a less than or equal to type constraint to convert the constraint into an equality. In economic terms, slack variables represent left-over or unused capacity.

Specifically:
$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ...+ a_{in}x_n \leq b_i$ can be written as
$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ...+ a_{in}x_n + s_i = b_i$

Where i = 1, 2, ..., m

Surplus variable

It is a variable subtracted from the left-hand side of a greater than or equal to type constraint to convert the constraint into an equality. It is also known as negative slack variable. In economic terms, surplus variables represent over fulfillment of the requirement.

Specifically:
$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \ldots\ldots\ldots + a_{in}x_n \geq b_i$ can be written as
$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \ldots\ldots\ldots + a_{in}x_n - s_i = b_i$

Where i = 1, 2, ..., m

Artificial variable

It is a non negative variable introduced to facilitate the computation of an initial basic feasible solution. In other words, a variable added to the left-hand side of a greater than or equal to type constraint to convert the constraint into an equality is called an artificial variable.

Simplex Method - Maximization Case

Consider the general linear programming problem

Maximize $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots + c_nx_n$

Subject to

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n \leq b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n \leq b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots + a_{mn}x_n \leq b_m$
$x_1, x_2, \ldots\ldots, x_n \geq 0$

Where:
c_j (j = 1, 2, ..., n) in the objective function are called the cost or profit coefficients.
b_i (i = 1, 2, ..., m) are called resources.
a_{ij} (i = 1, 2, ..., m; j = 1, 2, ..., n) are called technological coefficients or input-output coefficients.

Converting inequalities to equalities

Introducing slack variables to convert inequalities to equalities

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n + s_1 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n + s_2 = b_2$

....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ...+ a_{mn}x_n + s_m = b_m$
$x_1, x_2,....., x_n \geq 0$
$s_1, s_2,....., s_m \geq 0$

An initial basic feasible solution is obtained by setting $x_1 = x_2 =........ = x_n = 0$
$s_1 = b_1$
$s_2 = b_2$
....
$s_m = b_m$

The initial simplex table is formed by writing out the coefficients and constraints of a LPP in a systematic tabular form. The following table shows the structure of a simplex table.

Structure of a simplex table

	c_j	c_1	c_2	c_3	---	c_n	
c_B	**Basic variables B**	**x_1**	**x_2**	**x_3**	**---**	**x_n**	**Solution values b (=X_B)**
c_{B1}	x_1	a_{11}	a_{12}	a_{13}	---	a_{1n}	b_1
c_{B2}	x_2	a_{21}	a_{22}	a_{23}	---	a_{2n}	b_2
c_{B3}	x_3	a_{31}	a_{32}	a_{33}	---	a_{3n}	b_3
---	-----	----	----	-----	---	----	-----
c_{Bm}	x_n	a_{m1}	a_{m2}	a_{m3}	---	a_{mn}	b_m
z_j-c_j		z_1-c_1	z_2-c_2	z_3-c_3	---	z_n-c_n	

Where:
c_j = coefficients of the variables (m + n) in the objective function.
c_B = coefficients of the current basic variables in the objective function.
B = basic variables in the basis.
X_B = solution values of the basic variables.
z_j-c_j = index row.

The Simplex Algorithm

Unquestionably, the simplex technique has proved to be the most effective in solving linear programming problems. In the simplex method, we first find an initial basic solution (extreme point). Then, we proceed to an adjacent extreme point. We continue this process until we reach an optimal solution.

Steps (Maximization Case)

1. Formulate the Problem

- Formulate the mathematical model of the given linear programming problem.
- If the objective function is given in minimization form then convert it into maximization form in the following way:
  Min z = - Max (-z)

- Convert every inequality constraint in the L.P. problem into an equality constraint by adding a slack variable to each constraint.

2. Find out the Initial Solution

- Calculate the initial basic feasible solution by assigning zero value to the decision variables. This solution is shown in the initial simplex table.

3. Test for Optimality

- Calculate the values of zj – cj
- If the values of $z_j – c_j$ are positive, the current basic feasible solution is the optimal solution. If there are one or more negative values, choose the variable corresponding to which the value of $z_j – c_j$ is least (most negative) as this is likely to increase the profit most.

4. Test for Feasibility

- Divide the values under X_B column by the corresponding positive coefficient (a_{ij}) in the key column, and compare the ratios. The row that indicates the minimum ratio is called the key row. **However, division by zero or negative coefficients in the key column is not allowed**. In the case of a tie, break the tie arbitrarily.

5. Identify the Pivot Element (Key Element)

- The number that lies at the intersection of the key column and key row of a given table is called the key element. It is always a non-zero positive number.

6. Determine the New Solution

- The numbers in the replacing row may be obtained by dividing the key row elements by the pivot element. The numbers in the remaining rows may be calculated by using the following formula:

$$\text{New number} = \text{old number} - \frac{(\text{corresponding no. of key row}) \times (\text{corresponding no. of key column})}{\text{pivot element}}$$

7. Revise the Solution

- Go to step 3 and repeat the procedure until all the values of $z_j - c_j$ are either zero or positive.

Simplex Method - Examples

Do you know how to divide, multiply, add, and subtract? Yes. Then there is a good news for you. About 50% of this technique you already know.

Example 1

Maximize $z = 3x_1 + 2x_2$

Subject to

$-x_1 + 2x_2 \le 4$
$3x_1 + 2x_2 \le 14$
$x_1 - x_2 \le 3$

$x_1, x_2 \ge 0$

Solution

First, convert every inequality constraints in the LPP into an equality constraint, so that the problem can be written in a standard from. This can be accomplished by adding a slack variable to each constraint. Slack variables are always added to the less than type constraints.

Converting inequalities to equalities

$-x_1 + 2x_2 + x_3 = 4$
$3x_1 + 2x_2 + x_4 = 14$
$x_1 - x_2 + x_5 = 3$
$x_1, x_2, x_3, x_4, x_5 \ge 0$

Where x_3, x_4 and x_5 are slack variables.

Since slack variables represent unused resources, their contribution in the objective function is zero. Including these slack variables in the objective function, we get

Maximize $z = 3x_1 + 2x_2 + 0x_3 + 0x_4 + 0x_5$

Initial basic feasible solution

Now we assume that nothing can be produced. Therefore, the values of the decision variables are zero.
$x_1 = 0$, $x_2 = 0$, $z = 0$

When we are not producing anything, obviously we are left with unused capacity
$x_3 = 4$, $x_4 = 14$, $x_5 = 3$

We note that the current solution has three variables (slack variables x_3, x_4 and x_5) with non-zero solution values and two variables (decision variables x_1 and x_2) with zero values. **Variables with non-zero values are called basic variables. Variables with zero values are called non-basic variables.**

Table 1

	c_j	3	2	0	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	Solution values b (=X_B)
0	x_3	-1	2	1	0	0	4
0	x_4	3	2	0	1	0	14
0	x_5	1	-1	0	0	1	3
z_j-c_j		-3	-2	0	0	0	

$a_{11} = -1$, $a_{12} = 2$, $a_{13} = 1$, $a_{14} = 0$, $a_{15} = 0$, $b_1 = 4$
$a_{21} = 3$, $a_{22} = 2$, $a_{23} = 0$, $a_{24} = 1$, $a_{25} = 0$, $b_2 = 14$
$a_{31} = 1$, $a_{32} = -1$, $a_{33} = 0$, $a_{34} = 0$, $a_{35} = 1$, $b_3 = 3$

WORKINGS

Calculating values for the index row ($z_j – c_j$)

$z_1 – c_1 = (0 X (-1) + 0 X 3 + 0 X 1) - 3 = -3$
$z_2 – c_2 = (0 X 2 + 0 X 2 + 0 X (-1)) - 2 = -2$
$z_3 – c_3 = (0 X 1 + 0 X 0 + 0 X 0) - 0 = 0$
$z_4 – c_4 = (0 X 0 + 0 X 1 + 0 X 0) - 0 = 0$
$z_5 – c_5 = (0 X 0 + 0 X 0 + 0 X 1) – 0 = 0$

Choose the smallest negative value from $z_j - c_j$ (i.e., – 3). So column under x_1 is the key column.
Now find out the minimum positive value
Minimum (14/3, 3/1) = 3
So row x_5 is the key row.
Here, the pivot (key) element = 1 (the value at the point of intersection).
Therefore, x_5 departs and x_1 enters.

We obtain the elements of the next table using the following rules:

1. If the values of $z_j - c_j$ are positive, the inclusion of any basic variable will not increase the value of the objective function. Hence, the present solution maximizes the objective function. If there are more than one negative values, we choose the variable as a basic variable corresponding to which the value of $z_j - c_j$ is least (most negative) as this will maximize the profit.
2. The numbers in the replacing row may be obtained by dividing the key row elements by the pivot element and the numbers in the other two rows may be calculated by using the formula:

$$\text{New number} = \text{old number} - \frac{(\text{corresponding no. of key row}) \times (\text{corresponding no. of key column})}{\text{pivot element}}$$

Calculating values for table 2

x_3 row

$a_{11} = -1 - 1 \text{ X } ((-1)/1) = 0$
$a_{12} = 2 - (-1) \text{ X } ((-1)/1) = 1$
$a_{13} = 1 - 0 \text{ X } ((-1)/1) = 1$
$a_{14} = 0 - 0 \text{ X } ((-1)/1) = 0$
$a_{15} = 0 - 1 \text{ X } ((-1)/1) = 1$
$b_1 = 4 - 3 \text{ X } ((-1)/1) = 7$

x_4 row

$a_{21} = 3 – 1 X (3/1) = 0$
$a_{22} = 2 – (-1) X (3/1) = 5$
$a_{23} = 0 – 0 X (3/1) = 0$
$a_{24} = 1 – 0 X (3/1) = 1$
$a_{25} = 0 – 1 X (3/1) = -3$
$b_2 = 14 – 3 X (3/1) = 5$

x_1 row

$a_{31} = 1/1 = 1$
$a_{32} = -1/1 = -1$
$a_{33} = 0/1 = 0$
$a_{34} = 0/1 = 0$
$a_{35} = 1/1 = 1$
$b_3 = 3/1 = 3$

Table 2

	c_j	3	2	0	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	Solution values b (= X_B)
0	x_3	0	1	1	0	1	7
0	x_4	0	5	0	1	-3	5
3	x_1	1	-1	0	0	1	3
z_j-c_j		0	-5	0	0	3	

WORKINGS

Calculating values for the index row ($z_j – c_j$)

$z_1 – c_1 = (0 X 0 + 0 X 0 + 3 X 1) - 3 = 0$
$z_2 – c_2 = (0 X 1 + 0 X 5 + 3 X (-1)) – 2 = -5$
$z_3 – c_3 = (0 X 1 + 0 X 0 + 3 X 0) - 0 = 0$
$z_4 – c_4 = (0 X 0 + 0 X 1 + 3 X 0) - 0 = 0$
$z_5 – c_5 = (0 X 1 + 0 X (-3) + 3 X 1) – 0 = 3$

Key column = x_2 column
Minimum (7/1, 5/5) = 1
Key row = x_4 row

Pivot element = 5
x_4 departs and x_2 enters.

Calculating values for table 3

x_3 row

$a_{11} = 0 – 0 X (1/5) = 0$
$a_{12} = 1 – 5 X (1/5) = 0$
$a_{13} = 1 – 0 X (1/5) = 1$
$a_{14} = 0 – 1 X (1/5) = -1/5$
$a_{15} = 1 – (-3) X (1/5) = 8/5$
$b_1 = 7 – 5 X (1/5) = 6$

x_2 row

$a_{21} = 0/5 = 0$
$a_{22} = 5/5 = 1$
$a_{23} = 0/5 = 0$
$a_{24} = 1/5$
$a_{25} = -3/5$
$b_2 = 5/5 = 1$

x_1 row

$a_{31} = 1 – 0 X (-1/5) = 1$
$a_{32} = -1 – 5 X (-1/5) = 0$
$a_{33} = 0 – 0 X (-1/5) = 0$
$a_{34} = 0 – 1 X (-1/5) = 1/5$
$a_{35} = 1 – (-3) X (-1/5) = 2/5$
$b_3 = 3 – 5 X (-1/5) = 4$

Final Table

	c_j	3	2	0	0	0	
c_B	**Basic variables B**	x_1	x_2	x_3	x_4	x_5	**Solution values b (= X_B)**
0	x_3	0	0	1	-1/5	8/5	6
2	x_2	0	1	0	1/5	-3/5	1
3	x_1	1	0	0	1/5	2/5	4
z_j-c_j		0	0	0	1	0	

Since all the values of $z_j - c_j$ are positive, **this is the optimal solution.**
$x_1 = 4$, $x_2 = 1$
$z = 3 \times 4 + 2 \times 1 = 14$.

The largest profit of Rs.14 is obtained, when 1 unit of x_2 and 4 units of x_1 are produced. The above solution also indicates that 6 units are still unutilized, as shown by the slack variable x_3 in the X_B column.

Minimization Case

In the previous section, the simplex method was applied to linear programming problems where the objective was to maximize the profit with less than or equal to type constraints. In many cases, however, constraints may of type ≥ or = and the objective may be minimization (e.g., cost, time, etc.). Thus, in such cases, simplex method must be modified to obtain an optimal policy.

Consider the general linear programming problem

Minimize $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots + c_nx_n$

subject to

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n \geq b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n \geq b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots\ldots + a_{mn}x_n \geq b_m$
$x_1, x_2, \ldots.., x_n \geq 0$

Changing the sense of the optimization

Any linear minimization problem can be viewed as an equivalent linear maximization problem, and vice versa.

Min. $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots + c_nx_n$

It can be written as
Max. $z = -(c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots + c_nx_n)$

Converting inequalities to equalities

Introducing surplus variables (negative slack variables) to convert inequalities to equalities

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ...+ a_{1n}x_n - s_1 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ...+ a_{2n}x_n - s_2 = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ...+ a_{mn}x_n - s_m = b_m$
$x_1, x_2,....., x_n \geq 0$
$s_1, s_2,....., s_m \geq 0$

An initial basic feasible solution is obtained by setting $x_1 = x_2 =........ = x_n = 0$
$-s_1 = b_1$ or $s_1 = -b_1$
$-s_2 = b_2$ or $s_2 = -b_2$
....
$-s_m = b_m$ or $s_m = -b_m$

Which is not feasible because it violates the non-negativity stipulation, (i.e., $s_1 \geq 0$)? Therefore, we need artificial variables.

After introducing artificial variables, the set of constraints can be written as

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ...+ a_{1n}x_n - s_1 + A_1 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ...+ a_{2n}x_n - s_2 + A_2 = b_2$
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ...+ a_{mn}x_n - s_m + A_m = b_m$

$x_1, x_2,....., x_n \geq 0$
$s_1, s_2,....., s_m \geq 0$
$A_1, A_2,....., A_m \geq 0$

Now, an initial basic feasible solution can be obtained by setting all the decision and surplus variables to zero. Thus, an initial basic feasible solution to LPP is
$A_1 = b_1$, $A_2 = b_2$, ..., $A_m = b_m$

Now to obtain an optimal solution, we must drive out the artificial variables. Following are the two methods to solve linear programming problems in such cases.

- Two Phase method
- M method

Two Phase Method

In this method, the whole procedure of solving a linear programming problem involving artificial variables is divided into two phases. In phase I, we form a new objective function by assigning zero to every original variable (including slack and surplus variables) and -1 to each of the artificial variables. Then we try to eliminate the artificial variables from the basis. The solution at the end of phase I serves as a basic feasible solution for phase II. In phase II, the original objective function is introduced and the usual simplex algorithm is used to find an optimal solution.

Example 1

Minimize $z = -3x_1 + x_2 - 2x_3$

Subject to

$x_1 + 3x_2 + x_3 \leq 5$
$2x_1 - x_2 + x_3 \geq 2$
$4x_1 + 3x_2 - 2x_3 = 5$

$x_1, x_2, x_3 \geq 0$

Solution

Changing the sense of the optimization

Any linear minimization problem can be viewed as an equivalent linear maximization problem, and vice versa. Specifically:

Minimize $\sum_{j=1}^{n} c_j x_j$ = Maximize $\sum_{j=1}^{n} (-c_j) x_j$
If z is the optimal value of the left-hand expression, then -z is the optimal value of the right-hand expression.

Maximize $z = 3x_1 - x_2 + 2x_3$

Subject to

$x_1 + 3x_2 + x_3 \leq 5$
$2x_1 - x_2 + x_3 \geq 2$
$4x_1 + 3x_2 - 2x_3 = 5$

$x_1, x_2, x_3 \geq 0$

Converting inequalities to equalities

$x_1 + 3x_2 + x_3 + x_4 = 5$
$2x_1 – x_2 + x_3 – x_5 = 2$
$4x_1 + 3x_2 - 2x_3 = 5$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Where:
x_4 is a slack variable
x_5 is a surplus variable

Now, if we let x_1, x_2 and x_3 equal to zero in the initial solution, we will have $x_4 = 5$ and $x_5 = -2$, which is not possible because a surplus variable cannot be **negative**. Therefore, we need **artificial variables**.

$x_1 + 3x_2 + x_3 + x_4 = 5$
$2x_1 – x_2 + x_3 – x_5 + A_1 = 2$
$4x_1 + 3x_2 - 2x_3 + A_2 = 5$

$x_1, x_2, x_3, x_4, x_5, A_1, A_2 \geq 0$

Where A_1 and A_2 are artificial variables?

Phase 1

In this phase, we remove the artificial variables and find an initial feasible solution of the original problem. Now the objective function can be expressed as

Maximize $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + (–A_1) + (–A_2)$

Subject to

$x_1 + 3x_2 + x_3 + x_4 = 5$
$2x_1 – x_2 + x_3 – x_5 + A_1 = 2$
$4x_1 + 3x_2 - 2x_3 + A_2 = 5$

$x_1, x_2, x_3, x_4, x_5, A_1, A_2 \geq 0$

Initial basic feasible solution

The intial basic feasible solution is obtained by setting
$x_1 = x_2 = x_3 = x_5 = 0$

Then we shall have $A_1 = 2$, $A_2 = 5$, $x_4 = 5$

Table 1

	c_j	0	0	0	0	0	-1	-1	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	A_1	A_2	Solution values b (= X_B)
0	x_4	1	3	1	1	0	0	0	5
-1	A_1	2	-1	1	0	-1	1	0	2
-1	A_2	4	3	-2	0	0	0	1	5
$z_j - c_j$		-6	-2	1	0	1	0	0	

Key column = x_1 column
Minimum (5/1, 2/2, 5/4) = 1
Key row = A_1 row
Pivot element = 2
A1 departs and x_1 enters.

Table 2

	c_j	0	0	0	0	0	-1	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	A_2	Solution values b (= X_B)
0	x_4	0	7/2	1/2	1	1/2	0	4
0	x_1	1	-1/2	1/2	0	-1/2	0	1
-1	A_2	0	5	-4	0	2	1	1
$z_j - c_j$		0	-5	4	0	-2	0	

A2 departs and x_2 enters.
Here, Phase 1 terminates because both the artificial variables have been removed from the basis.

Phase 2

The basic feasible solution at the end of **Phase 1** computation is used as the initial basic feasible solution of the problem. The original objective function is introduced in **Phase 2** computation and the usual simplex procedure is used to solve the problem.

Table 3

	c_j	3	-1	2	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	Solution values b (= X_B)
0	x_4	0	0	33/10	1	-9/10	33/10
3	x_1	1	0	1/10	0	-3/10	11/10
-1	x_2	0	1	-4/5	0	2/5	1/5
z_j-c_j		0	0	-9/10	0	-13/10	

Table 4

	c_j	3	-1	2	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	Solution values b (= X_B)
0	x_4	0	9/4	3/2	1	0	15/4
3	x_1	1	3/4	-1/2	0	0	5/4
0	x_5	0	5/2	-2	0	1	1/2
z_j-c_j		0	13/4	-7/2	0	0	

Table 5

	c_j	3	-1	2	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	x_5	Solution values b (= X_B)
2	x_3	0	3/2	1	2/3	0	5/2
3	x_1	1	3/2	0	1/3	0	5/2
0	x_5	0	11/2	0	4/3	1	11/2
z_j-c_j		0	17/2	0	7/3	0	

An optimal policy is $x_1 = 5/2$, $x_2 = 0$, $x_3 = 5/2$. The associated optimal value of the objective function is z = 3 X (5/2) – 0 + 2 X (5/2) = 25/2.

The Big M Method

It is a modified version of the simplex method, in which we assign a very large value (M) to each of the artificial variables. We illustrate this method with the help of following examples.

Example 1

Maximize $z = x_1 + 5x_2$

subject to

$3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution.

Converting inequalities to equalities

By introducing surplus variables, slack variables and artificial variables, the standard form of LPP becomes

Maximize $x_1 + 5x_2 + 0x_3 + 0x_4 - MA_1$

subject to

$3x_1 + 4x_2 + x_3 = 6$
$x_1 + 3x_2 - x_4 + A_1 = 2$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, A_1 \geq 0$

Where:
x_3 is a slack variable
x_4 is a surplus variable.
A_1 is an artificial variable.

Initial basic feasible solution

$x_1 = x_2 = x_4 = 0$
$A_1 = 2, x_3 = 6$

Table 1

	c_j	1	5	0	0	–M	
c_B	Basic variables B	x_1	x_2	x_3	x_4	A_1	Solution values b (= X_B)
0	x_3	3	4	1	0	0	6
–M	A_1	1	3	0	–1	1	2
$z_j–c_j$		–M–1	–3M–5	0	M	0	

WORKINGS

Calculating values for index row ($z_j – c_j$)

$z_1 – c_1 = 0 X 3 + (–M) X 1 – 1 = –M – 1$
$z_2 – c_2 = 0 X 4 + (–M) X 3 – 5 = –3M – 5$
$z_3 – c_3 = 0 X 1 + (–M) X 0 – 0 = 0$
$z_4 – c_4 = 0 X 0 + (–M) X (–1) – 0 = M$
$z_5 – c_5 = 0 X 0 + (–M) X 1 – (–M) = 0$

As M is a large positive number, the coefficient of M in the $z_j–c_j$ row would decide the incoming basic variable.
As -3M < -M, x_2 becomes a basic variable in the next iteration.
Key column = x_2 column.
Minimum (6/4, 2/3) = 2/3
Key row = A1 row
Pivot element = 3.
A1 departs and x_2 enters.

Note that in the iteration just completed, artificial variable A_1 was eliminated from the basis. The new solution is shown in the following table.

Table 2

	c_j	1	5	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (= X_B)
0	x_3	5/3	0	1	4/3	10/3
5	x_2	1/3	1	0	–1/3	2/3
$z_j–c_j$		2/3	0	0	–5/3	

Table 3

	c_j	1	5	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (= X_B)
0	x_4	5/4	0	3/4	1	5/2
5	x_2	3/4	1	1/4	0	3/2
$z_j–c_j$		11/4	0	5/4	0	

The optimal solution is:

$x_1 = 0$, $x_2 = 3/2$
$z = 0 + 5 \text{ X } 3/2 = 15/2$

Some Special Cases

1. Unrestricted (unconstrained) Variables

Sometimes decision variables are unrestricted in sign (positive, negative or zero). In all such cases, the decision variables can be expressed as the difference between two non-negative variables. For example, if x_1 is unrestricted in sign, then

Put $x_1 = x_1' - x_1''$

Example

Maximize $z = 2x_1 + 3x_2$

subject to

$-x_1 + 2x_2 \leq 4$
$x_1 + x_2 \leq 6$
$x_1 + 3x_2 \leq 9$

x_1, x_2 are unrestricted in sign

Solution.

Since x_1 and x_2 are unrestricted in sign, we can replace them by non-negative variables x_1', x_1'', x_2', x_2''.

Put $x_1 = x_1' - x_1''$
$x_2 = x_2' - x_2''$

The given problem can be written as

Max. $z = 2(x_1' - x_1'') + 3(x_2' - x_2'')$

subject to

$-(x_1' - x_1'') + 2(x_2' - x_2'') \le 4$
$(x_1' - x_1'') + (x_2' - x_2'') \le 6$
$(x_1' - x_1'') + 3(x_2' - x_2'') \le 9$

Introducing slack variables

Max. $z = 2x_1' - 2x_1'' + 3x_2' - 3x_2''$

subject to

$-x_1' + x_1'' + 2x_2' - 2x_2'' + x_3 = 4$
$x_1' - x_1'' + x_2' - x_2'' + x_4 = 6$
$x_1' - x_1'' + 3x_2' - 3x_2'' + x_5 = 9$

Where x_3, x_4 and x_5 are slack variables

Table 1

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values b $(=X_B)$
0	x_3	-1	1	2	-2	1	0	0	4
0	x_4	1	-1	1	-1	0	1	0	6
0	x_5	1	-1	3	-3	0	0	1	9
z_j-c_j		-2	2	-3	3	0	0	0	

Key column = x_2' column.
Minimum (4/2, 6/1, 9/3) = 2
Key row = x_3 row.
Pivot element = 2
x_3 departs and x_2' enters.

Table 2

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values b (=X_B)
3	x_2'	-1/2	1/2	1	-1	1/2	0	0	2
0	x_4	3/2	-3/2	0	0	-1/2	1	0	4
0	x_5	5/2	-5/2	0	0	-3/2	0	1	3
z_j-c_j		-7/2	7/2	0	0	3/2	0	0	

Table 3

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values b (=X_B)
3	x_2'	0	0	1	-1	1/5	0	1/5	13/5
0	x_4	0	0	0	0	2/5	1	-3/5	11/5
2	x_1'	1	-1	0	0	-3/5	0	2/5	6/5
z_j-c_j		0	0	0	0	-3/5	0	7/5	

Table 4

	c_j	2	-2	3	-3	0	0	0	
c_B	Basic variables B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	x_5	Solution values b (=X_B)
3	x_2'	0	0	1	-1	0	-1/2	1/2	3/2
0	x_3	0	0	0	0	1	5/2	-3/2	11/2
2	x_1'	1	-1	0	0	0	3/2	-1/2	9/2
z_j-c_j		0	0	0	0	0	3/2	1/2	

The optimal solution is:
$x_1' = 9/2, x_1'' = 0, x_2' = 3/2, x_2'' = 0.$

Solution of the original problem is:
$x_1 = x_1' - x_1'' = 9/2 - 0 = 9/2$
$x_2 = x_2' - x_2'' = 3/2 - 0 = 3/2$
$z = 2 \text{ X } 9/2 + 3 \text{ X } 3/2 = 27/2.$

Self Test Questions

Theory

1. Write the general mathematical formulation of a linear programming problem.

2. Define the following:

- Slack variable
- Surplus variable
- Artificial variable

3. What do you mean by an optimal basic feasible solution to a linear programming problem?

4. Explain various steps of the simplex method involved in the computation of an optimal solution to a linear programming problem.

5. Fill up the blanks:

i. ….. variables are introduced to make ……… type inequalities equations.
ii. A basic solution with m equation and n variables has …… variables equal to zero.
iii. A basic feasible solution is a basic solution whose variables are …..
iv. The maximum number of basic feasible solutions in a system with m equations and n variables is ……
v. In a linear programming problem every …. point of the Convex set of feasible solutions is a ….solution of the problem.
vi. The objective function of a linear programming problem is maximized or minimized at a …. solution.

Practical

1. Maximize z = 5x + 3y

Subject to the constraints:

$x + y \leq 2$
$5x + 2y \leq 10$
$3x + 8y \leq 12$

$x, y \geq 0$

2. Maximize $z = 2x_1 + 4x_2 + x_3 + x_4$

Subject to

$x_1 + 3x_2 + x_4 \leq 4$
$2x_1 + x_2 \leq 3$
$x_2 + 4x_3 + x_4 \leq 3$

$x_1, x_2, x_3, x_4 \geq 0$

3. Maximize $z = 2x + 5y$

Subject to

$x + y \leq 600$
$0 \leq x \leq 400$
$0 \leq y \leq 300$

4. Maximize $z = 5x_1 + 3x_2$

Subject to

$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

5. Maximize $z = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$
$2x_1 - x_3 \leq 2$
$2x_1 - 2x_2 + 3x_3 \leq 0$

$x_1, x_2, x_3 \geq 0$

6. Maximize $z = x_1 - 3x_2 + 2x_3$

Subject to

$3x_1 - x_2 + 2x_3 \leq 7$
$-2x_1 + 4x_2 \leq 12$
$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

7. Maximize $z = -2x_1 - x_2$

Subject to

$x_1 + 2x_2 + x_3 = 10$
$x_1 + x_2 + x_4 = 6$
$x_1 - x_2 + x_5 = 2$
$x_1 - 2x_2 + x_6 = 1$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

8. Minimize $z = x_1 + x_2 + 3x_3$

Subject to

$3x_1 + 2x_2 + x_3 \leq 3$
$2x_1 + x_2 + 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

9. Minimize $z = x_2 - 3x_3 + 2x_5$

Subject to

$x_1 + 3x_2 - x_3 + 2x_5 = 7$
$-2x_2 + 4x_3 + x_4 = 12$
$-4x_2 + 3x_3 + 8x_5 + x_6 = 10$

$x_1, x_2, x_3 \geq 0$

10. Maximize $z = 2x_1 + 5x_2 + 7x_3$

Subject to

$3x_1 + 2x_2 + 4x_3 \leq 100$
$x_1 + 4x_2 + 2x_3 \leq 100$
$x_1 + x_2 + 3x_3 \leq 100$

$x_1, x_2, x_3 \geq 0$

11. Maximize $z = 6x + 5y - 3z - 4w$

Subject to

$2x + 3y + 2z - 4w = 24$
$x + 2y \leq 10$
$x + y + 2z + 3w \leq 15$
$y + z + w \leq 8$

$x, y, z, w \geq 0$

12. Maximize $z = 5x - 2y + 3z$

Subject to

$2x + 2y - z \geq 2$
$3x - 4y \leq 3$
$y + 3z \leq 5$

$x,y,z \geq 0$

13. Maximize $z = 8x_1 + 19x_2 + 7x_3$

Subject to

$3x_1 + 4x_2 + x_3 \leq 25$
$x_1 + 3x_2 + 3x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

14. Maximize: $z = x_1 + x_2 + x_3$

Subject to

$4x_1 + 5x_2 + 3x_3 \leq 15$
$10x_1 + 7x_2 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

15. Maximize: $z = 3x_1 + 4x_2$

Subject to

$x_1 - x_2 \leq 1$
$-x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

16. Maximize $z = 3x_1 + 5x_2 + 4x_3$

Subject to

$2x_2 + 3x_3 \leq 18$
$2x_2 + 5x_3 \leq 18$
$3x_1 + 2x_2 + 4x_3 \leq 25$

$x_1, x_2, x_3 > 10$

17. Maximize $z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 40$
$x_1 + x_2 \leq 24$
$2x_1 + 3x_2 \leq 60$

$x_1, x_2 \geq 0$

18. Maximize $z = 2x_1 + 4x_2$

Subject to

$2x_1 + 3x_2 \leq 48$
$x_1 + 3x_2 \leq 42$
$x_1 + x_2 \leq 21$

$x_1, x_2 \geq 0$

19. Minimize $z = 4x_1 + 8x_2 + 3x_3$

Subject to

$x_1 + x_2 \geq 2$
$2x_1 + x_3 \geq 5$

$x_1, x_2, x_3 \geq 0$

20. Minimize $z = x_1 + x_2 + x_3$

Subject to

$x_1 - x_4 - 2x_6 = 5$
$x_2 + 2x_4 - 3x_5 + x_6 = 3$
$x_3 + 2x_4 - 5x_5 + 6x_6 = 5$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

21. Minimize $z = 2x_1 + 9x_2 + x_3$

Subject to

$x_1 + 4x_2 + 2x_3 \geq 5$
$3x_1 + x_2 + 2x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

22. Minimize $z = 10x + 12y$

Subject to

$2x + 5y \geq 150$
$3x + y \geq 120$

$x, y \geq 0$

23. Maximize $z = 12x_1 + 15x_2 + 9x_3$

Subject to

$8x_1 + 16x_2 + 12x_3 \leq 250$
$4x_1 + 8x_2 + 10x_3 \geq 80$
$7x_1 + 9x_2 + 8x_3 = 105$

$x_1, x_2, x_3 \geq 0$

24. Maximize $z = 4x_1 + 14x_2$

Subject to

$2x_1 + 7x_2 \leq 21$
$7x_1 + 2x_2 \leq 21$

$x_1, x_2 \geq 0$

25. Maximize $z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2$
$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

26. Maximize $z = x_1 + x_2$

Subject to

$x_1 + x_2 \leq 1$
$-3x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

27. Maximize $z = 3x_1 + 2x_2$

Subject to

$x_1 - x_2 \leq 1$
$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

28. Consider the constraints

$-x_1 + x_2 \leq 1$
$6x_1 + 4x_2 \geq 24$

$x_1 \geq 0, x_{2 \geq 2.}$

(a) Minimize x_1.
(b) Minimize x_2.
(c) Maximize x_1.
(d) Maximize x_2.
(e) Minimize $x_1 + x_2$.
(f) Maximize $x_1 + x_2$.
(g) Maximize $-x_1 + 2x_2$.
(h) Maximize $x_1 - 2x_2$.
(i) Maximize $-3x_1 -2x_2$.

29. Consider the constraints

$-10x_1 - 15x_2 \geq -150$
$5x_1 + 10x_2 \geq 50$
$x_1 - x_2 \geq 0$

$x_1 \geq 2, x_{2 \geq 0.}$

(a) Maximize $x_1 + x_2$.
(b) Minimize $x_1 + x_2$.
(c) Maximize $x_1 + 3x_2$.
(d) Maximize $-2x_1 + x_2$.
(e) Maximize $-x_1 - 3x_2$.
(f) Maximize $-x_1 - 2x_2$.

Chapter - 4

Duality and Sensitivity Testing

Introduction

Duality is a very important concept associated with linear programming. The term 'Duality' implies that every linear programming problem, whether of maximization or minimization, is associated with another linear programming problem based on the same data. The original problem in this context is called the **primal problem**, whereas the other is called its **dual problem**. The formulation of the dual linear programming is sometimes referred to as duality. The notion of duality will deepen your understanding of what is really happening in the simplex method.

Why Duality must be studied?

Duality is an interesting feature of linear programming. Following are some reasons why duality must be studied.

- If the primal problems contain larger number of rows (constraints) and smaller number of columns (variables), converting it into dual can reduce the computational burden.
- Calculation of the dual checks the accuracy of the primal problem.
- It establishes the interconnections for all of the sensitivity analysis (post optimality analysis) techniques.
- It yields a number of powerful theorems, which add substantially to our understanding of linear programming approach.

We introduce the concept of duality with the help of following example.

Example

Minimize $z = 3x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 40$
$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

Solution.

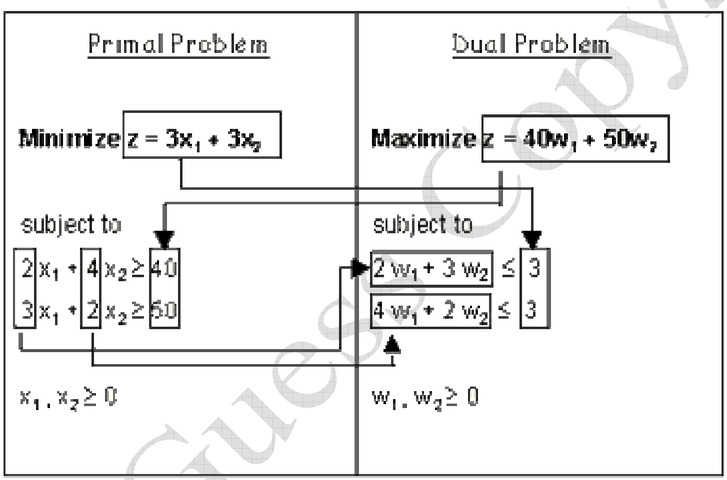
Maximize $z = 40w_1 + 50w_2$

subject to

$2w_1 + 3w_2 \leq 3$
$4w_1 + 2w_2 \leq 3$

$w_1, w_2 \geq 0$

The Primal-Dual construction relationship



Relationship between Primal and Dual Problem

- The number of constraints in the primal problem is equal to the number of dual variables, and *vice versa*.
- If the primal problem is a maximization problem, then the dual problem is a minimization problem and *vice versa*.
- If the primal problem has greater than or equal to type constraints, then the dual problem has less than or equal to type constraints and *vice versa*.
- The profit coefficients of the primal problem appear on the right-hand side of the dual problem.
- The rows in the primal become columns in the dual and *vice versa*..

Mixed Constraints

Example

Maximize $z = x_1 + 5x_2$

Subject to

$3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution

The above problem can be written as

Maximize $z = x_1 + 5x_2$

$3x_1 + 4x_2 \leq 6$
$-x_1 - 3x_2 \leq -2$

Now, the dual model of the problem can be formulated as follows:

Minimize $z = 6w_1 - 2w_2$

Subject to

$3w_1 - w_2 \geq 1$
$4w_1 - 3w_2 \geq 5$

$w_1, w_2 \geq 0$

Unrestricted Constraints

Example

Maximize $z = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 = 6$
$x_1 + 2x_2 + 5x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Solution

Converting equalities to inequalities

Any linear equality can be written as a set of like-directioned linear inequalities by imposing one additional constraint. The equation may be replaced by two weak inequalities. For instance, $x = 10$ can be written as $x \leq 10$ and $x \geq 10$, which in turn can be written as $x \leq 10$ and $-x \leq -10$.

So the constraints can be written as

$4x_1 + 3x_2 + x_3 \geq 6$
$4x_1 + 3x_2 + x_3 \leq 6$
$x_1 + 2x_2 + 5x_3 \geq 4$
$x_1 + 2x_2 + 5x_3 \leq 4$

Further, the above constraints can be written as

$-4x_1 - 3x_2 - x_3 \leq -6$
$4x_1 + 3x_2 + x_3 \leq 6$
$-x_1 - 2x_2 - 5x_3 \leq -4$
$x_1 + 2x_2 + 5x_3 \leq 4$

Now, the dual model of the problem can be formulated as follows:

Minimize $z = -6w_1 + 6w_2 - 4w_3 + 4w_4$

$-4w_1 + 4w_2 - w_3 + w_4 \geq 2$
$-3w_1 + 3w_2 - 2w_3 + 2w_4 \geq 3$
$-w_1 + w_2 - 5w_3 + 5w_4 \geq 1$

$w_1, w_2, w_3, w_4 \geq 0.$

Simplifying the problem

Let $y_1 = w_2 - w_1$, $y_2 = w_4 - w_3$

Minimize $z = 6y_1 + 4y_2$

Subject to
$4y_1 + y_2 \geq 2$
$3y_1 + 2y_2 \geq 3$
$y_1 + 5y_2 \geq 1$

y_1, y_2 are unrestricted.

The above problem can be directly solved by using the following rules:

Primal	Dual
i^{th} relation an equality	i^{th} variable unrestricted in sign
j^{th} variable unrestricted in sign	j^{th} relation an equality

Minimize $z = 6w_1 + 4w_2$

Subject to
$4w_1 + w_2 \geq 2$
$3w_1 + 2w_2 \geq 3$
$w_1 + 5w_2 \geq 1$

w_1, w_2 are unrestricted.

Dual Simplex Method

The Dual Simplex method is used in situations where the optimality criterion (i.e., $z_j - c_j \geq 0$ in the maximization case and $z_j - c_j \leq 0$ in minimization case) is satisfied, but the basic solution is not feasible because under the X_B column of the simplex table there are one or more negative values.

What are the reasons for studying the dual simplex method?

- Sometimes it allows to easily select an initial basis without having to add any artificial variable.
- It aids in certain types of sensitivity testing.
- It helps in solving integer programming problems.

Dual Simplex Method

Example

Minimize $z = 80x_1 + 100x_2$

Subject to

$80x_1 + 60x_2 \geq 1500$
$20x_1 + 90x_2 \geq 1200$

$x_1, x_2 \geq 0$

Solution

Minimize $z = 80x_1 + 100x_2$

Multiplying the constraints by -1 on both sides
$-80x_1 - 60x_2 \leq -1500$
$-20x_1 - 90x_2 \leq -1200$

After adding **slack variables**, the constraints are
$-80x_1 - 60x_2 + x_3 = -1500$
$-20x_1 - 90x_2 + x_4 = -1200$

Where x_3 and x_4 are slack variables?

Initial basic feasible solution

Substituting $x_1 = 0$, $x_2 = 0$, $z = 0$

This gives the solution values as
$x_3 = -1500$, $x_4 = -1200$

The initial simplex table is exhibited below.

Table 1

	c_j	80	100	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (=X_B)
0	x_3	-80	-60	1	0	-1500
0	x_4	-20	-90	0	1	-1200
z_j-c_j		-80	-100	0	0	

Key Row :
Min { X_{B1}, X_{B2}} = Min (-1500, -1200) = -1500
So x_3 row is the key row.

Key Column:

$$\textbf{Min} \left\{ \left| \frac{z_1 - c_1}{a_{11}} \right| , \left| \frac{z_2 - c_2}{a_{12}} \right| \right\}$$

$$\textbf{Min} \left\{ \left| \frac{-80}{-80} \right| , \left| \frac{-100}{-60} \right| \right\}$$

Min(1, 5/3) = 1
So column under x_1 becomes the key column.
Pivot element = - 80.
Therefore, x_3 departs & x_1 enters.

Table 2

	c_j	80	100	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (=X_B)
80	x_1	1	**3/4**	-1/80	0	75/4
0	x_4	**0**	-75	**-1/4**	**1**	**-825**
z_j-c_j		0	**-40**	-80	0	

Key row = x_4 row as it has the only negative value under the X_B column.
Key column = x_2 column.
Pivot element = -75.
x_4 departs & x_2 enters.

Table 3

	c_j	80	100	0	0	
c_B	Basic variables B	x_1	x_2	x_3	x_4	Solution values b (=X_B)
80	x_1	1	0	-3/200	1/100	21/2
100	x_2	0	1	1/300	-1/75	11
z_j-c_j		0	0	-13/15	-8/15	

The values for x_1 & x_2 are 21/2 & 11 respectively.
The associated objective function value is
z = 80 X 21/2 + 100 X 11 = 1940.

Algorithm - Steps

1. Formulate the Problem

- Formulate the **mathematical model** of the given linear programming problem.
- Convert every inequality constraint in the LPP into an equality constraint, so that the problem can be written in a standard from.

2. Find out the Initial Solution

- Calculate the initial basic feasible solution by assigning zero value to the decision variables. This solution is shown in the initial dual simplex table.

3. Determine an improved solution

- If all the values under X_B column ≥ 0, then don't apply dual simplex method because optimal solution can be easily obtained by the simplex method.
  On the contrary, if any value under X_B column < 0, then the current solution is infeasible so move to step 4.

4. Determine the key row

- Select the smallest (most) negative value under the X_B column. The row that indicates the smallest negative value is the key row.

5. Determine the key column

- Select the values of the non basic variables in the index row ($z_j - c_j$), and divide these values by the corresponding values of the key row determined in the previous step. Specifically,

$$\textbf{Key column} = \textbf{Min} \left\{ \left| \frac{z_j - c_j}{a_{ij}} \right| : \quad a_{ij} < 0 \right\}$$

7. Revise the Solution

- If all basic variables have non-negative values, an optimal solution has been obtained. If there are basic variables having negative values, then go to step 3.

Sensitivity (Post-optimality) Analysis

In linear programming, all model parameters are assumed to be constant; but in real life situations, the decision environment is always dynamic. Therefore, it is important for the management to know how profit would be affected by an increase or decrease in the resource level, by a change in the technological process, and by a change in the cost of raw materials. Such an investigation is known as sensitivity analysis or post-optimality analysis. The results of sensitivity analysis establish upper and lower bounds for input parameter values within which they can vary without causing violent changes in the current optimal solution.

The mechanics of sensitivity testing are explained with the help of following example.

Example

Luminous Lamps produces three types of lamps - A, B, and C. These lamps are processed on three machines - X, Y, and Z. The full technology and input restrictions are given in the following table.

Product	Machine			Profit per unit
	M1	M2	M3	
A	10	7	2	12

B	2	3	4	3
C	1	2	1	1
Available Time	100	77	80	

Solution.

The linear programming model for this problem can stated as:

Maximize $z = 12x_1 + 3x_2 + x_3$

subject to

$10x_1 + 2x_2 + x_3 \le 100$
$7x_1 + 3x_2 + 2x_3 \le 77$
$2x_1 + 4x_2 + x_3 \le 80$

$x_1, x_2, x_3 \ge 0$

The optimal solution to this problem is given below.

Final Table

	c_j	**12**	**3**	**1**	**0**	**0**	**0**	
c_B	**Basic variables B**	x_1	x_2	x_3	x_4	x_5	x_6	**Solution values b (=** X_B**)**
12	x_1	1	0	-1/16	3/16	-1/8	0	73/8
3	x_2	0	1	13/16	-7/16	5/8	0	35/8
0	x_6	0	0	-17/8	11/8	-9/4	1	177/4
z_j		12	3	27/16	15/16	3/8	0	
z_j-c_j		0	0	11/16	15/16	3/8	0	

An optimal policy is x_1 =73/8, x_2 = 35/8, x_3 = 0.
The associated optimal value of the objective function is 981/8.

Changes In Contribution Rate

First we investigate whether a previously determined optimal solution remains optimal if the contribution rate is changed. An increase in c_j of a variable would mean that resources from other products should be diverted to this more profitable product. The reverse is true for a minimization problem.

Changes in c_j of a non basic variable

A non basic variable can be brought into the basis only if its contribution rate becomes attractive. Hence, we need to determine the upper limit of the profit contribution (c_j) of each non basic variable. The reverse is true for a minimization problem.

From the above final simplex table, we note that profit contribution for product C is Re 1, which is not greater than its z_j. Thus, to bring x_3 into the basis, its profit contribution rate c_j must exceed Rs. 27/16 to make z_j-c_j value negative or zero (i.e., $z_j - c_j \leq 0$)

Specifically:

If $c_{j*} - c_j > (z_j - c_j)$, then a new optimal solution must be derived.
If $c_{j*} - c_j = (z_j - c_j)$, then alternative optimal solutions exist
If $c_{j*} - c_j < (z_j - c_j)$, then current optimal solution remains unchanged.

In our case $c_3 = 1$ and $z_3 - c_3 = 11/16$, then

$c_3^* - 1 \geq 11/16$
$c_3^* \geq 11/16 + 1 = 27/16$

x_3 can be introduced into the basis if its contribution rate c_3 increases upto atleast Rs. 27/16. If it increases beyond that then the current solution will no longer be optimal.

Change in c_j of a Basic Variable

Let us consider the case of product A (x_1 column), and divide each z_j-c_j entry in the index row (for non basic variable) by the corresponding coefficients in the x_1 row as shown below.

$$- \text{Minimum } (z_j - c_j / y_{1j};\ y_{1j} > 0) \leq \Delta 1 \leq \text{Minimum } (z_j - c_j / -y_{1j};\ y_{1j} < 0)$$

Referring to the final simplex table, we observe that corresponding to the non basic variables x_3 & x_5, $y_{13}, y_{15} < 0$ Hence,

$$\text{Minimum} \left[\frac{11/16}{-(-1/16)} , \frac{3/8}{-(-1/8)} \right]$$

= Minimum (11, 3) = 3

Corresponding to the non basic variable x_4, $y_{14} > 0$. Hence,

$$\text{Minimum}\left[\frac{15/16}{3/16}\right]$$

= 5

Hence,

$-5 \le c_1^* - 12 \le 3$, i.e., $7 \le c_1^* \le 15$

Thus, the optimal solution is insensitive so long as the changed profit coefficient c_1^* varies between Rs. 7 and Rs. 15.

Change In Available Resources

Now we investigate whether a previous optimal solution remains feasible if the available resources change. For long-term planning it is important to know the bounds within which each available resource (e.g., machine hours) can vary without causing violent changes in the current optimal solution. To illustrate, divide each quantity in the X_B column by the corresponding coefficient in the x_4 column of table.

X_B	x_4	X_B / x_4
73/8	3/16	146/3
35/8	-7/16	-10
177/4	11/8	354/11

The least positive ratio (354/11) indicates to how the number of hours of machine M_1 can be decreased. The least negative ratio (-10) indicates to how much the number of hours of machine M_1 can be increased.

Calculating the range

Lower limit = 100 - 354/11 = 746/11
Upper Limit = 100 - (-10) = 110.

Hence, the range of hours for M_1 is 746/11 to 110. By the same way, the range of hours for Machine M_2 & M3 can be calculated.

Change In Technological Coefficients

Changes in the technological coefficients reflect potential change either in the efficiency of manpower or in the state of technology. To illustrate, x_3 is non basic in the optimal solution of this example. Consider the possibility that its coefficient in second constraint is altered by Δ. Then, the associated dual is

$W_1 + (2 + \Delta)W_2 + W_3 \geq 1$

Substituting the current optimal values of the dual variables from the final simplex table.
$15/16 + (2 + \Delta) \times 3/8 + 0 \geq 1$
or $27 + 6\Delta \geq 16$
or $\Delta \geq - 11/6$

Therefore, if Δ is smaller than - 11/6, you should enter x_3 into the basis.

Self Test Questions

Theory

1. Define the dual of linear programming problem.

2. Explain the primal-dual relationship.

3. Write the algorithm of the dual simplex method.

4. Discuss differences between the standard simplex method and the dual simplex method.

5. How the dual can be useful in management decision making? Discuss.

6. Write the advantages of duality.

Practical

Use duality to solve the following LP problems.

1. Minimize $z = 3x_1 + x_2$

Subject to

$2x_1 + 3x_2 \geq 2$
$x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

2. Minimize $z = 50x_1 - 80x_2 - 140x_3$

Subject to

$x_1 - x_2 - 3x_3 \geq 4$
$x_1 - 2x_2 - 2x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$

3. Maximize $z = 3x_1 + 4x_2$

Subject to

$2x_1 + 3x_2 \leq 16$
$5x_1 + 2x_2 \geq 20$

$x_1, x_2 \geq 0$

4. Minimize $z = x_1 - x_2$

Subject to

$2x_1 - x_2 \geq 2$
$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

5. Maximize $z = x_1 + 5x_2$

Subject to

$3x_1 + 4x_2 \leq 6$
$x_1 + 3x_2 \leq 2$

$x_1, x_2 \geq 0$

6. Maximize $z = x_1 + x_2$

Subject to

$2x_1 + x_2 \geq 4$
$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

7. Minimize $z = 10y_1 + 6y_2 + 2y_3$

$-y_1 + y_2 + y_3 \geq 1$
$3y_1 + y_2 - y_3 \geq 2$

$y_1, y_2, y_3 \geq 0$

8. Minimize $z = 50x_1 - 80x_2 - 140x_3$

Subject to

$x_1 - x_2 - 3x_3 \geq 4$
$x_1 - 2x_2 - 2x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$

9. Minimize $z = 8x_1 - 2x_2 - 4x_3$

Subject to

$x_1 - 4x_2 - 2x_3 \geq 2$
$x_1 + x_2 - 3x_3 \geq -1$

$x_1, x_2, x_3 \geq 0$

10. Minimize $z = (15/2)x_1 - 3x_2$

Subject to

$3x_1 - x_2 - x_3 \geq 3$
$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

11. Minimize $z = x_3 + x_4 + x_5$

Subject to

$x_1 - x_3 + x_4 - x_5 = -2$
$x_2 - x_3 - x_4 + x_5 = 1$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

12. Minimize $z = x_1 + x_2 + x_3$

Subject to

$x_1 - 3x_2 + 4x_3 = 5$
$x_1 - 2x_2 \leq 3$
$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, x_3 unrestricted.

13. Maximize $z = 6x_1 + 4x_2 + 6x_3 + x_4$

Subject to

$4x_1 + 4x_2 + 4x_3 + 8x_4 = 21$
$3x_1 + 17x_2 + 80x_3 + 2x_4 \leq 48$

$x_1, x_2 \geq 0$, x_3, x_4 unrestricted.

Chapter - 5

Transportation Problem

Introduction

In this chapter and the chapter that follows, we explore the special structure of network models. This chapter focuses on the problems of product distribution.

Definition

The transportation problem is a special type of linear programming problem, where the objective is to minimize the cost of distributing a product from a number of sources to a number of destinations.

The transportation problem deals with a special class of linear programming problems in which the objective is to transport a homogeneous product manufactured at several plants (origins) to a number of different destinations at a minimum total cost. The total supply available at the origin and the total quantity demanded by the destinations are given in the statement of the problem. The cost of shipping a unit of goods from a known origin to a known destination is also given. Our objective is to determine the optimal allocation that results in minimum total shipping cost.

Mathematical Representation Of Transportation Problem

A firm has 3 factories - A, E, and K. There are four major warehouses situated at B, C, D, and M. Average daily product at A, E, K is 30, 40, and 50 units respectively. The average daily requirement of this product at B, C, D, and M is 35, 28, 32, 25 units respectively. The transportation cost (in Rs.) per unit of product from each factory to each warehouse is given below:

	Warehouse				
Factory	**B**	**C**	**D**	**M**	**Supply**
A	6	8	8	5	30
E	5	11	9	7	40
K	8	9	7	13	50
Demand	35	28	32	25	

The problem is to determine a routing plan that minimizes total transportation costs.

	B	C	D	M	Supply
A	6 x_{11}	8 x_{12}	8 x_{13}	5 x_{14}	30
E	5 x_{21}	11 x_{22}	9 x_{23}	7 x_{24}	40
K	8 x_{31}	9 x_{32}	7 x_{33}	13 x_{34}	50
Demand	35	28	32	25	

Let x_{ij} = no. of units of a product transported from ith factory(i = 1, 2, 3) to jth warehouse (j = 1, 2, 3, 4).

It should be noted that if in a particular solution the x_{ij} value is missing for a cell, this means that nothing is shipped between factory and warehouse.

The problem can be formulated mathematically in the linear programming form as
Minimize = $6x_{11} + 8x_{12} + 8x_{13} + 5x_{14}$
$+ 5x_{21} + 11x_{22} + 9x_{23} + 7x_{24}$
$+ 8x_{31} + 9x_{32} + 7x_{33} + 13x_{34}$

Subject to
Capacity constraints
$x_{11} + x_{12} + x_{13} + x_{14} = 30$
$x_{21} + x_{22} + x_{23} + x_{24} = 40$
$x_{31} + x_{32} + x_{33} + x_{34} = 50$

Requirement constraints
$x_{11} + x_{21} + x_{31} = 35$
$x_{12} + x_{22} + x_{32} = 28$
$x_{13} + x_{23} + x_{33} = 32$
$x_{14} + x_{24} + x_{34} = 25$

$x_{ij} \geq 0$

The above problem has 7 constraints and 12 variables. **Since no. of variables is very high, simplex method is not applicable. Therefore, more efficient methods have been developed to solve transportation problems.**

The general mathematical model may be given as follows

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to

$$\sum_{j=1}^{n} x_{ij} \leq S_i \quad \text{for } i = 1,2, \ldots.., m \text{ (supply)}$$
$$\sum_{i=1}^{m} x_{ij} \geq D_j \quad \text{for } j = 1,2, \ldots.., n \text{ (demand)}$$

$x_{ij} \geq 0$

For a feasible solution to exist, it is necessary that total capacity equals total requirements.
Total supply = total demand.
Or $\sum a_i = \sum b_j$.

What is the underlying assumption?

- Only a single type of commodity is being shipped from an origin to a destination.
- Total supply is equal to the total demand.
  $$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$
- S_i (supply) and D_j (demand) are all positive integers.

Basic Terminology

In this section, we augment your operations research vocabulary with some new terms.

Origin

It is the location from which shipments are dispatched.

Destination

It is the location to which shipments are transported.

Unit Transportation cost

It is the cost of transporting one unit of the consignment from an origin to a destination.

Perturbation Technique

It is a method used for modifying a degenerate transportation problem, so that the degeneracy can be resolved.

Feasible Solution

A solution that satisfies the row and column sum restrictions and also the non-negativity restrictions is a feasible solution.

Basic Feasible Solution

A feasible solution of (m X n) transportation problem is said to be basic feasible solution, when the total number of allocations is equal to (m + n – 1).

Optimal Solution

A feasible solution is said to be optimal solution when the total transportation cost will be the minimum cost.

In the sections that follow, we will concentrate on algorithms for finding solutions to transportation problems.

Methods for finding an initial basic feasible solution:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

North West Corner Rule

The North West corner rule is a method for computing a basic feasible solution of a transportation problem, where the basic variables are selected from the North – West corner (i.e., top left corner).

The standard North West Corner Rule instructions are paraphrased below:

Steps

1. Select the north west (upper left-hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e., min(s1, d1).

2. Adjust the supply and demand numbers in the respective rows and columns.
3. If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.
4. If the supply for the first row is exhausted, then move down to the first cell in the second row.
5. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
6. Continue the process until all supply and demand values are exhausted.

Example 1

The Amulya Milk Company has three plants located throughout a state with production capacity 50, 75 and 25 gallons. Each day the firm must furnish its four retail shops R_1, R_2, R_3, & R_4 with at least 20, 20 , 50, and 60 gallons respectively. The transportation costs (in Rs.) are given below.

Plant	Retail Shop				Supply
	R_1	R_2	R_3	R_4	
P_1	3	5	7	6	50
P_2	2	5	8	2	75
P_3	3	6	9	2	25
Demand	20	20	50	60	

The economic problem is to distribute the available product to different retail shops in such a way so that the total transportation cost is minimum

Solution

Starting from the North West corner, we allocate min (50, 20) to P_1R_1, i.e., 20 units to cell P_1R_1. The demand for the first column is satisfied. The allocation is shown in the following table.

Table 1

Plant	Retail Shop				Supply
	R_1	R_2	R_3	R_4	
P_1	3 20	5	7	6	~~50~~ 30
P_2	2	5	8	2	75
P_3	3	6	9	2	25
Demand	~~20~~	20	50	60	

Now we move horizontally to the second column in the first row and allocate 20 units to cell P_1R_2. The demand for the second column is also satisfied.

Table 2

Plant	Retail Shop				Supply
	R_1	R_2	R_3	R_4	
P_1	3 (20)	5 (20)	7	6	~~50~~ ~~30~~ 10
P_2	2	5	8	2	75
P_3	3	6	9	2	25
Demand	~~20~~	~~20~~	50	60	

Proceeding in this way, we observe that $P_1R_3 = 10$, $P_2R_3 = 40$, $P_2R_4 = 35$, $P_3R_4 = 25$. The resulting feasible solution is shown in the following table.

Final Table

Plant	Retail Shop				Supply
	R_1	R_2	R_3	R_4	
P_1	3 (20)	5 (20)	7 (10)	6	~~50~~
P_2	2	5	8 (40)	2 (35)	~~75~~
P_3	3	6	9	2 (25)	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

Here, number of retail shops (n) = 4, and
Number of plants (m) = 3

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6.

Initial basic feasible solution

The total transportation cost is calculated by multiplying each x_{ij} in an occupied cell with the corresponding c_{ij} and adding as follows:

20 X 3 + 20 X 5 + 10 X 7 + 40 X 8 + 35 X 2 + 25 X 2 = 670

Example 2

Luminous lamps have three factories - F_1, F_2, and F_3 with production capacity 30, 50, and 20 units per week respectively. These units are to be shipped to four warehouses W_1, W_2,

W_3, and W_4 with requirement of 20, 40, 30, and 10 units per week respectively. The transportation costs (in Rs.) per unit between factories and warehouses are given below.

Factory	Warehouse				Supply
	W_1	W_2	W_3	W_4	
F_1	1	2	1	4	30
F_2	3	3	2	1	50
F_3	4	2	5	9	20
Demand	20	40	30	10	

Find an initial basic feasible solution of the given transportation problem

Solution

Starting from the North West corner, we allocate 20 units to F_1W_1. The demand for the first column is completely satisfied.

Table 1

Factory	Warehouse				Supply
	W_1	W_2	W_3	W_4	
F_1	1 (20)	2	1	4	~~30~~ 10
F_2	3	3	2	1	50
F_3	4	2	5	9	20
Demand	~~20~~	40	30	10	

Proceeding in this way, we observe that $F_1W_2 = 10$, $F_2W_2 = 30$, $F_2W_3 = 20$, $F_3W_3 = 10$, $F_3W_4 = 10$. An initial basic feasible solution is exhibited below.

Final Table

Factory	Warehouse				Supply
	W_1	W_2	W_3	W_4	
F_1	1 (20)	2 (10)	1	4	~~30~~
F_2	3	3 (30)	2 (20)	1	~~50~~
F_3	4	2	5 (10)	9 (10)	~~20~~
Demand	~~20~~	~~40~~	~~30~~	~~10~~	

Number of basic variables = $m + n - 1 = 3 + 4 - 1 = 6$.

Initial basic feasible solution

20 X 1 + 10 X 2 + 30 X 3 + 20 X 2 + 10 X 5 + 10 X 9 = 310.

Matrix Minimum Method

Matrix minimum (Least cost) method is a method for computing a basic feasible solution of a transportation problem, where the basic variables are chosen according to the unit cost of transportation. This method is very useful because it reduces the computation and the time required to determine the optimal solution. The following steps summarize the approach.

Steps

1. Identify the box having minimum unit transportation cost (c_{ij}).
2. If the minimum cost is not unique, then you are at liberty to choose any cell.
3. Choose the value of the corresponding x_{ij} as much as possible subject to the capacity and requirement constraints.
4. Repeat steps 1-3 until all restrictions are satisfied.

Example 1

Consider the transportation problem presented in the following table:

Factory	Retail Shop				Supply
	1	2	3	4	
1	3	5	7	6	50
2	2	5	8	2	75
3	3	6	9	2	25
Demand	20	20	50	60	

Solution.

We observe that c_{21} =2, which is the minimum transportation cost. So x_{21} = 20. The demand for the first column is satisfied. The allocation is shown in the following table.

Table 1

Factory	Retail Shop				Supply
	1	2	3	4	
1	3	5	7	6	50
2	2 (20)	5	8	2	~~75~~ 55
3	3	6	9	2	25
Demand	~~20~~	20	50	60	

Now we observe that c_{24} =2, which is the minimum transportation cost, so x_{24} = 55. The supply for the second row is exhausted.

Table 2

Factory	Retail Shop				Supply
	1	2	3	4	
1	3	5	7	6	50
2	2 (20)	5	8	2 (55)	~~75~~
3	3	6	9	2	25
Demand	~~20~~	20	50	~~60~~ 5	

Proceeding in this way, we observe that x_{34} = 5, x_{12} = 20, x_{13} = 30, x_{33} = 20. The resulting feasible solution is shown in the following table.

Final Table

Factory	Retail Shop				Supply
	1	2	3	4	
1	3	5 (20)	7 (30)	6	~~50~~
2	2 (20)	5	8	2 (55)	~~75~~
3	3	6	9 (20)	2 (5)	~~25~~
Demand	~~20~~	~~20~~	~~50~~	~~60~~	

Number of basic variables = m + n –1 = 3 + 4 – 1 = 6.

Initial basic feasible solution

The total transportation cost associated with this solution is calculated as given below:
20 X 2 + 20 X 5 + 30 X 7 + 55 X 2 + 20 X 9 + 5 X 2 = 650.

Example 2

Consider the transportation problem presented in the following table:

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	450

Solution.

We observe that $F_2W_2 = 8$, which is the minimum transportation cost and allocate 120 units to it. The demand for the second column is satisfied.

Table 1

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8 (120)	18	~~160~~ 40
F_3	26	24	16	90
Demand	180	~~120~~	150	450

The resulting feasible solution is shown in the following table.

Final Table

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16 50	20	12 150	~~200~~
F_2	14 40	8 120	18	~~160~~
F_3	26 90	24	16	~~90~~
Demand	~~180~~	~~120~~	~~150~~	450

Number of basic variables = m + n –1 = 3 + 3 – 1 = 5.

Initial basic feasible solution

The total transportation cost associated with this solution is calculated as given below:
50 X 16 + 150 X 12 + 40 X 14 + 120 X 8 + 90 X 26 = 6460.

Vogel Approximation Method

The Vogel approximation (Unit penalty) method is an iterative procedure for computing a basic feasible solution of a transportation problem. This method is preferred over the two methods discussed in the previous sections, because the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

Steps

The standard instructions are paraphrased below:

1. Identify the boxes having minimum and next to minimum transportation cost in each row and write the difference (penalty) along the side of the table against the corresponding row.
2. Identify the boxes having minimum and next to minimum transportation cost in each column and write the difference (penalty) against the corresponding column
3. Identify the maximum penalty. If it is along the side of the table, make maximum allotment to the box having minimum cost of transportation in that row. If it is below the table, make maximum allotment to the box having minimum cost of transportation in that column.
4. If the penalties corresponding to two or more rows or columns are equal, you are at liberty to break the tie arbitrarily.
5. Repeat the above steps until all restrictions are satisfied.

Example 1

Consider the transportation problem presented in the following table:

Destination					
Origin	**1**	**2**	**3**	**4**	**Supply**
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
Demand	60	40	30	110	240

Solution

Calculating penalty for table 1

17 - 4 = 13, 9 - 7 = 2, 20 - 15 = 5
24 - 20 = 4, 37 - 22 = 15, 17 - 9 = 8, 7 - 4 = 3

Table 1

Destination						
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**
1	20	40 22	17	4	~~120~~ 80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
Demand	60	~~40~~	30	110	240	
Penalty	4	15	8	3		

The highest penalty occurs in the second column. The minimum cij in this column is c_{12} (i.e., 22). So x_{12} = 40 and the second column is **eliminated**. The new reduced matrix is shown below:

Now again calculate the penalty.

Table 2

Origin	1	3	4	Supply	Penalty
1	20	17	4 (80)	~~80~~	13
2	24	9	7	70	2
3	32	20	15	50	5
Demand	60	30	110		
Penalty	4	8	3		

The highest penalty occurs in the first row. The minimum c_{ij} in this row is c_{14} (i.e., 4). So x_{14} = 80 and the first row is eliminated. The new reduced matrix is shown below:

Table 3

Origin	1	3	4	Supply	Penalty
2	24	9 (30)	7	70	2
3	32	20	15	50	5
Demand	60	~~30~~	30		
Penalty	8	11	8		

The highest penalty occurs in the second column. The minimum cij in this column is c_{23} (i.e., 9). So x_{23} = 30 and the second column is eliminated. The reduced matrix is given in the following table.

Table 4

Origin	1	4	Supply	Penalty
2	24 (10)	7 (30)	~~40~~	17
3	32 (50)	15	~~50~~	17
Demand	~~60~~	~~30~~		
Penalty	8	8		

The following table shows the computation of penalty for various rows and columns.

Final table

				Destination							
Origin	**1**	**2**	**3**	**4**	**Supply**	**Penalty**					
1	20	22 (40)	17	4 (80)	~~120~~	13	13	-	-	-	-
2	24 (10)	37	9 (30)	7 (30)	~~70~~	2	2	2	17	24	24
3	32 (50)	37	20	15	~~50~~	5	5	5	17	32	-
Demand	~~60~~	~~40~~	~~30~~	~~110~~	240						
Penalty	4	15	8	3							
	4	-	8	3							
	8	-	11	8							
	8	-	-	8							
	8	-	-	-							
	24	-	-	-							

Initial basic feasible solution

22 X 40 + 4 X 80 + 24 X 10 + 9 X 30 + 7 X 30 + 32 X 50 = 3520.

Stepping Stone Method

It is a method for finding the optimum solution of a transportation problem.

Steps

1. Determine an initial basic feasible solution using any one of the following:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

2. Make sure that the number of occupied cells is exactly equal to m+n-1, where m is the number of rows and n is the number of columns.

3. Select an unoccupied cell. Beginning at this cell, trace a closed path, starting from the selected unoccupied cell until finally returning to that same unoccupied cell.

4. Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, beginning with the plus sign at unoccupied cell to be evaluated.

5. Add the unit transportation costs associated with each of the cell traced in the closed path. This will give net change in terms of cost.

6. Repeat steps 3 to 5 until all unoccupied cells are evaluated.

7. Check the sign of each of the net change in the unit transportation costs. If all the net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost, so move to step 8..

8. Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to this cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

For clarity of exposition, consider the following transportation problem.

Example 1

A company has three factories A, B, and C with production capacity 700, 400, and 600 units per week respectively. These units are to be shipped to four depots D, E, F, and G with requirement of 400, 450, 350, and 500 units per week respectively. The transportation costs (in Rs.) per unit between factories and depots are given below:

	Depot				
Factory	**D**	**E**	**F**	**G**	**Capacity**
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

The decision problem is to minimize the total transportation cost for all factory-depot shipping patterns.

Solution.

An initial basic feasible solution is obtained by Matrix Minimum Method and is shown below in table 1.

Table 1

Depot					
Factory	D	E	F	G	Capacity
A	4	6 450	8	6 250	700
B	3 50	5	2 350	5	400
C	3 350	9	6	5 250	600
Requirement	400	450	350	500	1700

Here, m + n - 1 = 6. So the solution is not degenerate.

Initial basic feasible solution

6 X 450 + 6 X 250 + 3 X 50 + 2 X 350 + 3 X 350 + 5 X 250 = 7350

Table 2

The cell AD (4) is empty so allocate one unit to it. Now draw a closed path from AD. The result of allocating one unit along with the necessary adjustments in the adjacent cells is indicated in table 2.

Depot					
Factory	D	E	F	G	Capacity
A	+ 4 +1	6 450	8	- 6 249	700
B	3 50	5	2 350	5	400
C	- 3 349	9	6	+ 5 251	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is
+4 – 6 + 5 – 3 = 0.

This indicates that every unit allocated to route AD will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

Choose another unoccupied cell. The cell BE is empty so allocate one unit to it.
Now draw a closed path from BE as shown below in table 3.

Table 3

Depot					
Factory	D	E	F	G	Capacity
A	4	- 6 (449)	8	+ 6 (251)	700
B	- 3 (49)	+ 5 (+1)	2 (350)	5	400
C	+ 3 (351)	9	6	- 5 (249)	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is
+5 – 6 + 6 – 5 + 3 – 3 = 0

This indicates that every unit allocated to route BE will neither increase nor decrease the transportation cost. Thus, such a reallocation is unnecessary.

We must evaluate all such unoccupied cells in this manner by finding closed paths and calculating the net cost change as shown below.

Unoccupied cells	Increase in cost per unit of reallocation	Remarks
CE	+9 – 5 + 6 – 6 = 4	Cost Increases
CF	+6 – 3 + 3 – 2 = 4	Cost Increases
AF	+8 – 6 + 5 – 3 + 3 – 2 = 5	Cost Increases
BG	+5 – 5 + 3 – 3 = 0	Neither increase nor decrease

Since all the values of unoccupied cells are greater than or equal to zero, the solution obtained is optimal.

Minimum transportation cost is:
6 X 450 + 6 X 250 + 3 X 50 + 2 X 350 + 3 X 350 + 5 X 250 = Rs. 7350

Example 2

Consider the following transportation problem (cost in rupees)

Distributor				
Factory	**D**	**E**	**F**	**Supply**

A	2	1	5	10
B	7	3	4	25
C	6	5	3	20
Requirement	15	22	18	55

Find out the minimum cost of the given transportation problem.

Solution.

We compute an initial basic feasible solution of the problem by Matrix Minimum Method as shown in table 1.

Table 1

Distributor				
Factory	**D**	**E**	**F**	**Supply**
A	2	1 (10)	5	10
B	7 (13)	3 (12)	4	25
C	6 (2)	5	3 (18)	20
Requirement	15	22	18	55

Here, m + n - 1 = 5. So the solution is not degenerate.

Initial basic feasible solution

1 X 10 + 7 X 13 + 3 X 12 + 6 X 2 + 3 X 18 = 203

The cell AD (2) is empty so allocate one unit to it. Now draw a closed path.

Table 2

Distributor				
Factory	D	E	F	Supply
A	+ 2 (+1)	- 1 (9)	5	10
B	- 7 (12)	+ 3 (13)	4	25
C	6 (2)	5	3 (18)	20
Requirement	15	22	18	55

The increase in the transportation cost per unit quantity of reallocation is:
+ 2 – 1 + 3 – 7 = - 3.

The allocations for other unoccupied cells are following:

Unoccupied cells	Increase in cost per unit of reallocation	Remarks
AF	+5 – 1 + 3 – 7 + 6 – 3 = 3	Cost Increases
CE	+5 – 3 + 7 – 6 = 3	Cost Increases
BF	+4 – 7 + 6 – 3 = 0	Neither increase nor decrease

This indicates that the route through AD would be beneficial to the company. The maximum amount that can be allocated to AD is 10 and this will make the current basic variable corresponding to cell AE non basic.

Table 3 shows the transportation table after reallocation.

Table 3

Distributor				
Factory	**D**	**E**	**F**	**Supply**
A	2 (10)	1	5	10
B	7 (3)	3 (22)	4	25
C	6 (2)	5	3 (18)	20
Requirement	15	22	18	55

Since the reallocation in any other unoccupied cell cannot decrease the transportation cost, the minimum transportation cost is:
2 X 10 + 7 X 3 + 3 X 22 + 6 X 2 + 3 X 18 = Rs.173

Modified Distribution Method (MODI) or (u - v) method

The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem. In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Steps

1. Determine an initial basic feasible solution using any one of the three methods given below:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

2. Determine the values of dual variables, u_i and v_j, using $u_i + v_j = c_{ij}$

3. Compute the opportunity cost using $c_{ij} – (u_i + v_j)$.

4. Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand, if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.

5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.

6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, **add** this quantity to all the cells on

the corner points of the closed path marked with plus signs, and **subtract** it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

9. Repeat the whole procedure until an optimal solution is obtained.

Example

Consider the transportation problem presented in the following table.

Distribution centre						
		D1	**D2**	**D3**	**D4**	**Supply**
Plant	**P1**	19	30	50	12	7
	P2	70	30	40	60	10
	P3	40	10	60	20	18
Requirement		5	8	7	15	

Determine the optimal solution of the above problem.

Solution

An initial basic feasible solution is obtained by Matrix Minimum Method and is shown in table 1.

Table 1

Distribution centre						
		D1	**D2**	**D3**	**D4**	**Supply**
Plant	**P1**	19	30	50	12 7	7
	P2	70 3	30	40 7	60	10
	P3	40 2	10 8	60	20 8	18
Requirement		5	8	7	15	

Initial basic feasible solution

12 X 7 + 70 X 3 + 40 X 7 + 40 X 2 + 10 X 8 + 20 X 8 = Rs. 894.

WORKINGS

Calculating u_i and v_j using $u_i + v_j = c_{ij}$

Substituting $u_1 = 0$, we get

$u_1 + v_4 = c_{14} \Rightarrow 0 + v_4 = 12$ or $v_4 = 12$
$u_3 + v_4 = c_{34} \Rightarrow u_3 + 12 = 20$ or $u_3 = 8$
$u_3 + v_2 = c_{32} \Rightarrow 8 + v_2 = 10$ or $v_2 = 2$
$u_3 + v_1 = c_{31} \Rightarrow 8 + v_1 = 40$ or $v_1 = 32$
$u_2 + v_1 = c_{21} \Rightarrow u_2 + 32 = 70$ or $u_2 = 38$
$u_2 + v_3 = c_{23} \Rightarrow 38 + v_3 = 40$ or $v_3 = 2$

Table 2

Distribution centre							
		D1	**D2**	**D3**	**D4**	**Supply**	**u_i**
Plant	**P1**	19	30	50	12 (7)	7	0
	P2	70 (3)	30	40 (7)	60	10	38
	P3	40 (2)	10 (8)	60	20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

WORKINGS

Calculating opportunity cost using $c_{ij} – (u_i + v_j)$

Unoccupied cells	Opportunity cost
(P_1, D_1)	$c_{11} – (u_1 + v_1) = 19 – (0 + 32) = –13$
(P_1, D_2)	$c_{12} – (u_1 + v_2) = 30 – (0 + 2) = 28$
(P_1, D_3)	$c_{13} – (u_1 + v_3) = 50 – (0 + 2) = 48$
(P_2, D_2)	$c_{22} – (u_2 + v_2) = 30 – (38 + 2) = –10$
(P_2, D_4)	$c_{14} – (u_2 + v_4) = 60 – (38 + 12) = 10$
(P_3, D_3)	$c_{33} – (u_3 + v_3) = 60 – (8 + 2) = 50$

Table 3

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	-13 19	28 30	48 50	12 (7)	7	0
	P2	70 (3)	-10 30	40 (7)	10 60	10	38
	P3	40 (2)	10 (8)	50 60	20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Now choose the smallest (most) negative value from opportunity cost (i.e., –13) and draw a closed path from P1D1. The following table shows the closed path.

Table 4

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	-13 19 +	28 30	48 50	- 12 (7)	7	0
	P2	70 (3)	-10 30	40 (7)	10 60	10	38
	P3	40 (2) -	10 (8)	50 60	+ 20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Choose the smallest value with a negative position on the closed path(i.e., 2), it indicates the number of units that can be shipped to the entering cell. Now add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

Now again calculate the values for u_i & v_j and opportunity cost. The resulting matrix is shown below.

Table 5

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (2)	[28] 30	[61] 50	12 (5)	7	0
	P2	70 (3)	[-23] 30	40 (7)	[-3] 60	10	51
	P3	[13] 40	10 (8)	[63] 60	20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Choose the smallest (most) negative value from opportunity cost (i.e., –23). Now draw a closed path from P2D2 .

Table 6

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	+ 19 (2)	[28] 30	[61] 50	– 12 (5)	7	0
	P2	– 70 (3)	[-23] 30 +	40 (7)	[-3] 60	10	51
	P3	[13] 40	– 10 (8)	[63] 60	+ 20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Now again calculate the values for u_i & v_j and opportunity cost

Table 7

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (5)	[28] 30	[38] 50	12 (2)	7	0
	P2	[23] 70	30 (3)	40 (7)	[20] 60	10	28
	P3	[13] 40	10 (5)	[40] 60	20 (13)	18	8
Requirement		5	8	7	15		
v_j		19	2	12	12		

Since all the current opportunity costs are non–negative, this is the optimal solution. The minimum transportation cost is: 19 X 5 + 12 X 2 + 30 X 3 + 40 X 7 + 10 X 5 + 20 X 13 = Rs. 799

Degeneracy

If the basic feasible solution of a transportation problem with m origins and n destinations has fewer than m + n – 1 positive x_{ij} (occupied cells), the problem is said to be a degenerate transportation problem.

Degeneracy can occur at two stages:

1. At the initial solution
2. During the testing of the optimal solution

To resolve degeneracy, we make use of an artificial quantity (*d*). The quantity *d* is assigned to that unoccupied cell, which has the minimum transportation cost.

The use of *d* is illustrated in the following example.

Example

Factory	Dealer				Supply
	1	2	3	4	
A	2	2	2	4	1000
B	4	6	4	3	700
C	3	2	1	0	900
Requirement	900	800	500	400	

Solution

An initial basic feasible solution is obtained by Matrix Minimum Method.

Table 1

Factory	Dealer				Supply
	1	2	3	4	
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2	1 (500)	0 (400)	900
Requirement	900	800	500	400	

Number of basic variables = m + n – 1 = 3 + 4 – 1 = 6
Since number of basic variables is less than 6, therefore, it is a degenerate transportation problem.

To resolve degeneracy, we make use of an artificial quantity(d). The quantity d is assigned to that unoccupied cell, which has the minimum transportation cost.

In the above table, there is a tie in selecting the smallest unoccupied cell. In this situation, you can choose any cell arbitrarily. We select the cell C2 as shown in the following table.

Table 2

Factory	Dealer				Supply
	1	2	3	4	
A	2 (900)	2 (100)	2	4	1000
B	4	6 (700)	4	3	700
C	3	2 (d)	1 (500)	0 (400)	900 + d
Requirement	900	800 + d	500	400	2600 + d

Now, we use the stepping stone method to find an optimal solution.

Calculating opportunity cost

Unoccupied cells	Increase in cost per unit of reallocation	Remarks
A3	+2 – 2 + 2 – 1 = 1	Cost Increases
A4	+4 – 2 + 2 – 0 = 4	Cost Increases
B1	+4 – 6 + 2 – 2 = –2	Cost Decreases
B3	+4 – 6 + 2 – 1 = –1	Cost Decreases
B4	+3 – 6 + 2 – 0 = –1	Cost Decreases
C1	+3 – 2 + 2 – 2 = 1	Cost Increases

The cell B1 is having the maximum improvement potential, which is equal to -2. The maximum amount that can be allocated to B1 is 700 and this will make the current basic variable corresponding to cell B2 non basic. The improved solution is shown in the following table.

Table 3

Factory	Dealer				Supply
	1	2	3	4	
A	2 (200)	2 (800)	2	4	1000
B	4 (700)	6	4	3	700
C	3	2 (d)	1 (500)	0 (400)	900
Requirement	900	800	500	400	2600

The optimal solution is
2 X 200 + 2 X 800 + 4 X 700 + 2 X d + 1 X 500 + 0 X 400 = 5300 + 2d.
Notice that *d* is a very small quantity so it can be neglected in the optimal solution. Thus, the net transportation cost is Rs. 5300

Unbalanced Transportation Problem

So far we have assumed that the total supply at the origins is equal to the total requirement at the destinations.
Specifically,

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

But in certain situations, the total supply is not equal to the total demand. Thus, the transportation problem with unequal supply and demand is said to be unbalanced transportation problem.

How we can solve an unbalanced transportation problem?

If the total supply is more than the total demand, we introduce an additional column, which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply, an additional row is introduced in the table, which represents unsatisfied demand with transportation cost zero. The balancing of an unbalanced transportation problem is illustrated in the following example.

Example

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Solution:

The total demand is 1000, whereas the total supply is 800.

$$\sum_{i=1}^{m} S_i < \sum_{j=1}^{n} D_j$$

Total supply < total demand.
To solve the problem, we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200
Demand	250	250	500	1000

Using matrix minimum method, we get the following allocations.

Plant	Warehouse			Supply
	W1	W2	W3	
A	50 28	17	450 26	~~500~~
B	19	250 12	50 16	~~300~~
Unsatisfied demand	200 0	0	0	~~200~~
Demand	~~250~~	~~250~~	~~500~~	1000

Initial basic feasible solution

50 X 28 + 450 X 26 + 250 X 12 + 50 X 16 + 200 X 0 = 16900.

Maximization In A Transportation Problem

There are certain types of transportation problems where the objective function is to be maximized instead of being minimized. These problems can be solved by converting the maximization problem into a minimization problem.

Example

Surya Roshni Ltd. has three factories - X, Y, and Z. It supplies goods to four dealers spread all over the country. The production capacities of these factories are 200, 500 and 300 per month respectively.

Factory	Dealer				Capacity
	A	B	C	D	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand	180	320	100	400	

Determine a suitable allocation to maximize the total net return.

Solution:

Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost.

Here, the maximum transportation cost is 25. So subtract each value from 25. The revised transportation problem is shown below.

Table 1

Factory	Dealer				Capacity
	A	B	C	D	
X	13	7	19	0	200
Y	17	18	15	7	500
Z	11	22	14	5	300
Demand	180	320	100	400	

An initial basic feasible solution is obtained by matrix-minimum method and is shown in the final table.

Final table

Factory	Dealer				Capacity
	A	B	C	D	
X	13	7	19	0 (200)	~~200~~
Y	17 (80)	18 (320)	15 (100)	7	~~500~~
Z	11 (100)	22	14	5 (200)	~~300~~
Demand	~~180~~	~~320~~	~~100~~	~~400~~	

The maximum net return is

25 X 200 + 8 X 80 + 7 X 320 + 10 X 100 + 14 X 100 + 20 X 200 = 14280.

Prohibited Routes

Sometimes there may be situations, where it is not possible to use certain routes in a transportation problem. For example, road construction, bad road conditions, strike, unexpected floods, local traffic rules, etc. We can handle such type of problems in different ways:

- A very large cost represented by M or $\propto$ is assigned to each of such routes, which are not available.
- To block the allocation to a cell with a prohibited route, we can cross out that cell.

The problem can then be solved in its usual way.

Example

Consider the following transportation problem.

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16	$\propto$	12	200
F_2	14	8	18	160
F_3	26	$\propto$	16	90
Demand	180	120	150	450

Solution:

An initial solution is obtained by the matrix minimum method and is shown in the final table.

Final Table

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16 (50)	$\propto$	12 (150)	~~200~~
F_2	14 (40)	8 (120)	18	~~160~~
F_3	26 (90)	$\propto$	16	~~90~~
Demand	~~180~~	~~120~~	~~150~~	450

Initial basic feasible solution

16 X 50 + 12 X 150 + 14 X 40 + 8 X 120 + 26 X 90 = 6460.

The minimum transportation cost is Rs. 6460.

Time Minimizing Problem

Succinctly, it is a transportation problem in which the objective is to minimize the time. This problem is same as the transportation problem of minimizing the cost, expect that the unit transportation cost is replaced by the time t_{ij}.

Steps

1. Determine an initial basic feasible solution using any one of the following:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

2. Find T_k for this feasible plan and cross out all the unoccupied cells for which $t_{ij} \geq T_k$.

3. Trace a closed path for the occupied cells corresponding to T_k. If no such closed path can be formed, the solution obtained is optimum otherwise, go to step 2.

Example 1

The following matrix gives data concerning the transportation times t_{ij}

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25	30	20	40	45	37	37
O2	30	25	20	30	40	20	22
O3	40	20	40	35	45	22	32
O4	25	24	50	27	30	25	14
Demand	15	20	15	25	20	10	

Solution.

We compute an initial basic feasible solution by north west corner rule which is shown in table 1.

Table 1

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 (15)	30 (20)	20 (2)	40	45	37	37
O2	30	25	20 (13)	30 (9)	40	20	22
O3	40	20	40	35 (16)	45 (16)	22	32
O4	25	24	50	27	30 (4)	25 (10)	14
Demand	15	20	15	25	20	10	

Here, t_{11} = 25, t_{12} = 30, t_{13} = 20, t_{23} = 20, t_{24} = 30, t_{34}= 35, t_{35} = 45, t_{45} =30, t_{46} = 25

Choose maximum from t_{ij}, i.e., T_1 = 45. Now, cross out all the unoccupied cells that are ≥ T1.
The unoccupied cell (O3D6) enters into the basis as shown in table 2.

Table 2

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	- 45 16	+ 22	32
O4	25	24	~~50~~	27	+ 30 4	- 25 10	14
Demand	15	20	15	25	20	10	

Choose the smallest value with a negative position on the closed path, i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Table 3

Destination							
Origin	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**Supply**
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	30 9	40	20	22
O3	40	20	40	35 16	45 6	22 10	32
O4	25	24	~~50~~	27	30 14	25	14
Demand	15	20	15	25	20	10	

Here, T_2 = Max(25, 30, 20, 20, 20, 35, 45, 22, 30) = 45. Now, cross out all the unoccupied cells that are $\geq T_2$.

Table 4

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 15	30 20	20 2	40	~~45~~	37	37
O2	30	25	20 13	- 30 9	+ 40	20	22
O3	40	20	40	+ 35 16	- 45 6	22 10	32
O4	25	24	~~50~~	27	30 14	25	14
Demand	15	20	15	25	20	10	

By following the same procedure as explained above, we get the following revised matrix.

Table 6

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25 15	30 20	20 2	~~40~~	~~45~~	37	37
O2	30	25	20 13	30 3	40 6	20	22
O3	~~40~~	20	~~40~~	35 22	~~45~~	22 10	32
O4	25	24	~~50~~	27	30 14	25	14
Demand	15	20	15	25	20	10	

T_3 = Max(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40. Now, cross out all the unoccupied cells that are $\geq T_3$.

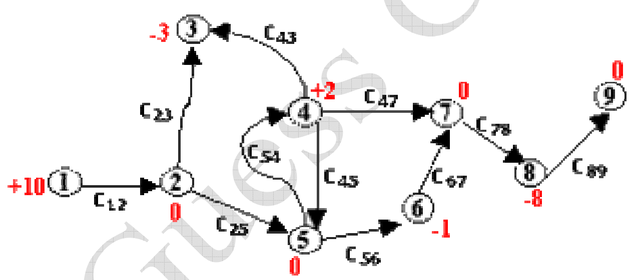
Now we cannot form any other closed loop with T_3.
Hence, the solution obtained at this stage is optimal.
Thus, all the shipments can be made within 40 units.

Transshipment Model

In a transportation problem, consignments are always transported from an origin to a destination. But, there could be several situations where it might be economical to transport items via one or more intermediate centers (or stages). In a transshipment problem, the available commodity is not sent directly from sources to destinations, i.e., it passes through one or more intermediate points before reaching the actual destination. For instance, a company may have regional warehouses that distribute the products to smaller district warehouses, which in turn ship to the retail stores. Succinctly, the transshipment model is an extension of the classical transportation model where an item available at point i is shipped to demand point j through one or more intermediate points.

Example

A company has nine large stores located in several states. The sales department is interested in reducing the price of a certain product in order to dispose all the stock now in hand. But, before that the management wants to reposition its stock among the nine stores according to its sales expectations at each location.



The above figure shows the numbered nodes (9 stores). A positive value next to a store represents the amount of inventory to be redistributed to the rest of the system. A negative value represents the shortage of stock. Thus, stores 1 and 4 have excess stock of 10 & 2 items respectively. Stores 3, 6 & 8 need 3, 1, and 8 more items respectively. The inventory positions of stores 2, 5 & 7 are to remain unchanged.

An item may be shipped through stores 2, 4, 5, 6, 7 & 8. These locations are known as transshipment points. Each remaining store is a **source** if it has excess stock, and a **sink** if it needs stock. In the above figure, store 1 is a source and store 3 is a sink.

The value c_{ij} is the cost of transporting items. To transport an item from store 1 to store 3, the total shipping cost is
$\mathbf{c_{12} + c_{23}}$

In the following example, you will learn how to convert a transshipment problem to a standard transportation problem.

Example

Consider a transportation problem where the origins are plants and destinations are depots. The unit transportation costs, capacity at the plants, and the requirements at the depots are indicated below:

Table 1

Plant	Depot			
	X	Y	Z	
A	1	3	15	150
B	3	5	25	300
	150	150	150	450

When each plant is also considered a destination and each depot is also considered an origin, there are altogether five origins and five destinations. Some additional cost data are also necessary. These are presented in the following Tables.

Table 2

Unit Transportation Cost from Plant to Plant		
From Plant	To	
	Plant A	Plant B
A	0	65
B	1	0

Table 3

Unit Transportation Cost from Depot to Depot			
From Depot	To		
	Depot X	Depot Y	Depot Z
X	0	23	1
Y	1	0	3
Z	65	3	0

Table 4

Unit Transportation Cost from Depot to Plant		
Depot	**Plant**	
	A	**B**
X	3	15
Y	25	3
Z	45	55

Solution.

From Table 1, Table 2, Table 3 and Table 4 we obtain the transportation formulation of the transshipment problem.

Table 5

Transshipment Table						
	A	**B**	**X**	**Y**	**Z**	**Capacity**
A	0	65	1	3	15	150 + 450 = 600
B	1	0	3	5	25	300 + 450 = 750
X	3	15	0	23	1	450
Y	25	3	1	0	3	450
Z	45	55	65	3	0	450
Requirement	450	450	150 + 450 =600	150 + 450 =600	150 + 450 =600	2700

The transportation model is extended and now it includes five supply points & demand points. To have a supply and demand from all the points, a fictitious supply and demand quantity (**buffer stock**) of 450 is added to both supply and demand of all the points. An initial basic feasible solution is obtained by the **Vogel's Approximation method** and is shown in the final table.

Final Table

Transshipment Table						
	A	**B**	**X**	**Y**	**Z**	**Capacity**
A	150 0	65	300 1	150 3	15	600
B	300 1	450 0	3	5	25	750
X	3	15	300 0	23	150 1	450
Y	25	3	1	450 0	3	450
Z	45	55	65	3	450 0	450
Requirement	450	450	600	600	600	2700

The total transhipment cost is:
0 X 150 + 1 X 300 + 3 X 150 + 1 X 300 + 0 X 450 + 0 X 300 + 1 X 150 + 0 X 450 + 0 X 450 = 1200

Self Test Questions

Theory

1. Describe the transportation problem with its general mathematical formulation.

2. Explain the following:

- North west corner rule
- Matrix minimum method
- Vogel approximation method

3. Explain the following with the help of an example:

- Unbalanced transportation problem
- Maximization in a transportation problem
- Degeneracy
- Prohibited routes

4. Discuss the similarities and differences between the stepping stone method and the MODI method in solving a transportation problem.

5. Fill in the blanks

i. The objective of a transportation problem is to …………………. the transportation cost.
ii. The constraints of a transportation problem are ………………….
iii. If a transportation problem has m factories and n retail shops the number of variables is …………………. and the number of constraints is …………………

Practical

1. A factory has three warehouses W1, W2 and W3 which supply to four stores S1, S2, S3 and S4. Monthly capacities of the warehouses are W1 = 100 units, W2 = 40 units and W3 = 60 units. Monthly demands at the stores are S1 = 30 units, S2 = 50 units, S3 = 65 units and S4= 55 units.
The shipping cost in terms of rupees from warehouses to stores is as given below:

	Stores			
Warehouses	**S1**	**S2**	**S3**	**S4**
W1	14	16	12	20
W2	12	14	10	8
W3	10	16	8	15

The problem here is to determine the optimum distribution for the factory to minimize shipping costs.

2. Solve the following transportation problems.

(a)

	Stores			
Factories	**1**	**2**	**3**	**Supply**
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

(b)

Stores							
Factories	**A**	**B**	**C**	**D**	**E**	**F**	**Supply**
1	1	2	1	4	5	1	30
2	3	3	2	1	4	3	50
3	4	2	5	9	6	2	75
4	3	1	7	3	4	6	20
Demand	20	40	30	10	50	25	

(c)

	X	**Y**	**Z**	a_i
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
b_j	50	80	80	

(d)

	D1	**D2**	**D3**	**D4**	a_i
O1	10	7	3	6	3
O2	1	6	8	3	5
O3	7	4	5	3	7
b_j	3	2	6	4	15

(e)

	D1	**D2**	**D3**	a_i
O1	7	3	3	4
O2	3	1	4	1
O3	4	3	6	5
b_j	2	3	5	10

(f)

	D1	D2	D3	D4	D5	D6	a_i
O1	9	12	9	6	9	10	5
O2	7	3	7	7	5	5	6
O3	6	5	9	11	3	11	2
O4	6	8	11	2	2	10	2
b_j	4	4	6	2	4	2	

(g)

	D1	D2	D3	D4	a_i
O1	4	3	0	5	24
O2	1	2	6	1	17
O3	3	6	2	3	19
b_j	15	19	18	8	60

3. Solve the following transportation problems by Stepping Stone Method and MODI method:

(a)

Distributor				
Factory	**1**	**2**	**3**	**Inventory**
1	2	1	5	10
2	7	3	4	25
3	6	5	3	20
Order	15	22	18	55

(b)

Plant	Market				Available
	A	**B**	**C**	**D**	
X	19	30	50	10	7
Y	70	30	40	60	9
Z	40	8	70	20	18
Required	5	8	7	14	

(c)

Plant	Market				Available
	A	B	C	D	
X	14	9	18	6	11
Y	10	11	7	16	13
Z	25	20	11	34	19
Required	6	10	12	15	

(d)

Plant	Market				Available
	A	B	C	D	
X	10	22	0	20	8
Y	15	20	12	8	13
Z	20	12	10	15	11
Required	5	11	8	8	

(e)

	D1	D2	D3	D4	D5	Total
O1	12	4	9	5	9	55
O2	8	1	6	6	7	45
O3	1	12	4	7	7	30
O4	10	15	6	9	1	50
Total	40	20	50	30	40	

(f)

	D1	D2	D3	Total
O1	2	4	1	40
O2	6	3	2	50
O3	4	5	6	20
O4	3	2	1	30
O5	5	2	5	10
Total	50	60	40	150

4. Solve the following transportation problem.

Furnaces	Mills					
	M1	M2	M3	M4	M5	Total
F1	4	2	3	2	6	8
F2	5	4	5	2	1	12
F3	6	5	4	7	3	14
Requirement	4	4	6	8	8	

Chapter - 6

Assignment Problem

Introduction

The assignment problem is a special type of transportation problem, where the objective is to minimize the cost or time of completing a number of jobs by a number of persons. In other words, when the problem involves the allocation of *n* different facilities to *n* different tasks, it is often termed as an assignment problem. The model's primary usefulness is for planning. The assignment problem also encompasses an important sub-class of so-called shortest- (or longest-) route models. The assignment model is useful in solving problems such as, assignment of machines to jobs, assignment of salesmen to sales territories, travelling salesman problem, etc.

It may be noted that with n facilities and n jobs, there are n! possible assignments. One way of finding an optimal assignment is to write all the n! possible arrangements, evaluate their total cost, and select the assignment with minimum cost. But, due to heavy computational burden this method is not suitable. This chapter concentrates on an efficient method for solving assignment problems that was developed by a **Hungarian mathematician D.Konig.**

Formulation of an assignment problem

Suppose a company has n persons of different capacities available for performing each different job in the concern, and there are the same number of jobs of different types. One person can be given one and only one job. The objective of this assignment problem is to assign n persons to n jobs, so as to minimize the total assignment cost. The cost matrix for this problem is given below:

Job					
Worker	j_1	j_2	----	j_n	a_i
i_1	x_{11} c_{11}	x_{12} c_{12}	----	x_{n1} c_{1n}	1
i_2	x_{21} c_{21}	x_{22} c_{22}	----	x_{2n} c_{2n}	1
----	----	----	----	----	--
i_n	x_{n1} c_{n1}	x_{n2} c_{n2}	----	x_{nn} c_{nn}	1
b_j	1	1	----	1	

To formulate the assignment problem in mathematical programming terms, we define the activity variables as

$$x_{ij} = \begin{cases} 1 \text{ if job } j \text{ is performed by worker } i \\ 0 \text{ otherwise} \end{cases}$$

for i = 1, 2, ..., n and j = 1, 2, ..., n

In the above table, c_{ij} is the cost of performing jth job by ith worker.

The optimization model is

Minimize $c_{11}x_{11} + c_{12}x_{12} + \text{-------} + c_{nn}x_{nn}$

Subject to

$x_{i1} + x_{i2} + ...+ x_{in} = 1$ $\quad$ i = 1, 2,......., n

$x_{1j} + x_{2j} + ...+ x_{nj} = 1$ $\quad$ j = 1, 2,......., n

$x_{ij} = 0$ or 1

In Σ Sigma notation

Minimize $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$

Subject to

$\sum_{i=1}^{n} x_{ij} = 1$ for i = 1, 2, ..., n

$\sum_{j=1}^{n} x_{ij} = 1$ for j = 1, 2, ..., n

$x_{ij} = 0$ or 1 for all i and j

Assumptions

- Number of jobs is equal to the number of machines or persons.
- Each man or machine is assigned only one job.
- Each man or machine is independently capable of handling any job to be done.
- Assigning criteria is clearly specified (minimizing cost or maximizing profit).

Hungarian Method

It is an efficient method for solving assignment problems. This method is based on the following principle:

- If a constant is added to, or subtracted from, every element of a row and/or a column of the given cost matrix of an assignment problem, the resulting assignment problem has the same optimal solution as the original problem.

Algorithm

The objective of this section is to examine a computational method - an algorithm - for deriving solutions to the assignment problems. The following steps summarize the approach:

Steps

1. Identify the minimum element in each row and subtract it from every element of that row.
2. Identify the minimum element in each column and subtract it from every element of that column.
3. Make the assignments for the reduced matrix obtained from **steps 1 and 2** in the following way:
   i. For each row or column with a single zero value cell that has not be assigned or eliminated, box ☐that zero value as an assigned cell.
   ii. For every zero that becomes assigned, cross out (X) all other zeros in the same row and the same column.

iii. If for a row and a column, there are two or more zeros and one cannot be chosen by inspection, then you are at liberty to choose the cell arbitrarily for assignment.
iv. The above process may be continued until every zero cell is either assigned ☐or crossed (X).

4. An optimal assignment is found, if the number of assigned cells equals the number of rows (and columns). In case you have chosen a zero cell arbitrarily, there may be alternate optimal solutions. If no optimal solution is found, go to step 5.
5. Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:
   i. Mark all the rows that do not have assignments.
   ii. Mark all the columns (not already marked) which have zeros in the marked rows.
   iii. Mark all the rows (not already marked) that have assignments in marked columns.
   iv. Repeat steps 5 (i) to (iii) until no more rows or columns can be marked.
   v. Draw straight lines through all unmarked rows and marked columns.
6. Select the smallest element from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.
7. Go to **step 3** and repeat the procedure until you arrive at an optimal assignment.

Hungarian Method - Examples

Now we will examine a few highly simplified illustrations. Later in the chapter, you will find more practical versions of assignment models like Crew assignment problem, Traveling salesman problem, etc.

Example 1

The Funny Toys Company has four men available for work on four separate jobs. Only one man can work on any one job. The cost of assigning each man to each job is given in the following table. The objective is to assign men to jobs in such a way that the total cost of assignment is minimum.

Job				
Person	**1**	**2**	**3**	**4**
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

Solution:

Step 1

Identify the minimum element in each row and **subtract** it from every element of that row. The result is shown in the following table.

Table

Job				
Person	**1**	**2**	**3**	**4**
A	0	5	2	8
B	0	3	8	2
C	2	0	4	7
D	2	0	1	1

Step 2

Identify the minimum element in each column and subtract it from every element of that column.

Table

Job				
Person	**1**	**2**	**3**	**4**
A	0	5	1	7
B	0	3	7	1
C	2	0	3	6
D	2	0	0	0

Step 3

Make the assignments for the reduced matrix obtained from **steps 1 and 2** in the following way:

a. For each row or column with a single zero value cell that has not be assigned or eliminated, box ☐that zero value as an assigned cell.
b. For every zero that becomes assigned, cross out (X) all other zeros in the same row and the same column.
c. If for a row and a column, there are two or more zeros and one cannot be chosen by inspection, choose the cell arbitrarily for assignment.
d. The above process may be continued until every zero cell is either assigned ☐or crossed (X).

Step 4

An optimal assignment is found, if the number of assigned cells equals the number of rows (and columns). In case you have chosen a zero cell arbitrarily, there may be alternate optimal solutions. If no optimal solution is found, go to step 5.

Table

	Job			
Person	**1**	**2**	**3**	**4**
A	$\boxed{0}$	5	1	7
B	$\cancel{0}$	3	7	1
C	2	$\boxed{0}$	3	6
D	2	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Step 5

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

i. Mark all the rows that do not have assignments.
ii. Mark all the columns (not already marked) which have zeros in the marked rows.
iii. Mark all the rows (not already marked) that have assignments in marked columns.
iv. Repeat steps 5 (ii) and (iii) until no more rows or columns can be marked.
v. Draw straight lines through all unmarked rows and marked columns.

Table

	Job			
Person	1	2	3	4
A	0	5	1	7
B	0	3	7	1
C	2	0	3	6
D	2	0	0	0

Step 6

Select the smallest element (i.e., 1) from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.

Table

	Job			
Person	1	2	3	4
A	0	4	0	6
B	0	2	6	0
C	3	0	3	6
D	3	0	0	0

Now again make the assignments for the reduced matrix.

Final Table

	Job			
Person	1	2	3	4
A	0	4	0	6
B	0	2	6	0
C	3	0	3	6
D	3	0	0	0

Since the number of assignments is equal to the number of rows (& columns), this is the optimal solution.

The total cost of assignment = A1 + B4 + C2 + D3

Substituting values from original table:
20 + 17 + 17 + 24 = Rs. 78.

Some Special Cases

1. Unbalanced Assignment Problem

In the previous section, the number of persons and the number of jobs were assumed to be the same. In this section, we remove this assumption and consider a situation where the number of persons is not equal to the number of jobs. In all such cases, fictitious rows and/or columns are added in the matrix to make it a square matrix. Then, we apply the usual Hungarian Method to this resulting balanced assignment problem. We provide the following example to illustrate the solution of an unbalanced assignment problem.

Example

	Job			
Person	**1**	**2**	**3**	**4**
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24

Solution

Since the number of persons is less than the number of jobs, we introduce a dummy person (D) with zero values. The revised assignment problem is given below:

Table

	Job			
Person	**1**	**2**	**3**	**4**
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D (dummy)	0	0	0	0

Now use the Hungarian method to obtain the optimal solution yourself.
Ans. = 20 + 17 + 17 + 0 = 54.

2. Maximization In An Assignment Problem

There are problems where certain facilities have to be assigned to a number of jobs, so as to maximize the overall performance of the assignment. The Hungarian Method can also solve such problems, as it is easy to obtain an equivalent minimization problem by converting every number in the matrix to an opportunity loss. The conversion is accomplished by subtracting all the elements of the given matrix from the highest element. It turns out that minimizing opportunity loss produces the same assignment solution as the original maximization problem.

Example

At the head office of www.universalteacher.com there are five registration counters. Five persons are available for service.

	Person				
Counter	**A**	**B**	**C**	**D**	**E**
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

How should the counters be assigned to persons so as to **maximize the profit**?

Solution:

Here, the highest value is **62**. So we subtract each value from 62. The conversion is shown in the following table.

Table

	Person				
Counter	**A**	**B**	**C**	**D**	**E**
1	32	25	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	33	0	21	28	23

Now the above problem can be easily solved by Hungarian method. After applying steps 1 to 3 of the Hungarian method, we get the following matrix.

Table

Person					
Counter	**A**	**B**	**C**	**D**	**E**
1	10	3	$\boxed{0}$	8	$\cancel{0}$
2	$\boxed{0}$	16	13	15	4
3	$\cancel{0}$	8	7	6	5
4	15	2	$\cancel{0}$	$\boxed{0}$	4
5	33	$\boxed{0}$	21	24	23

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix.

Table

Person					
Counter	A	B	C	D	E
1	10	3	$\boxed{0}$	8	$\cancel{0}$
2	$\boxed{0}$	16	13	15	4
3	$\cancel{0}$	8	7	6	5
4	15	2	$\cancel{0}$	$\boxed{0}$	4
5	33	$\boxed{0}$	21	24	23

Select the smallest element from all the uncovered elements, i.e., 4. Subtract this element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment. Repeating step 3, we obtain a solution which is shown in the following table.

Table

Person					
Counter	**A**	**B**	**C**	**D**	**E**
1	14	3	$\boxed{0}$	8	$\cancel{0}$
2	$\cancel{0}$	12	9	11	$\boxed{0}$
3	$\boxed{0}$	4	3	2	1
4	19	2	$\cancel{0}$	$\boxed{0}$	4
5	37	$\boxed{0}$	21	24	23

The total cost of assignment = 1C + 2E + 3A + 4D + 5B

Substituting values from original table:
40 + 36 + 40 + 36 + 62 = 214.

3. Multiple Optimal Solutions

Sometimes, it is possible to cross out all the zeros in the reduced matrix in two or more ways. If you can choose a zero cell arbitrarily, then there will be multiple optimum solutions with the same total pay-off for assignments made. In such a case, the management may select that set of optimal assignments, which is more suited to their requirement.

Example

The Spicy Spoon restaurant has four payment counters. There are four persons available for service. The cost of assigning each person to each counter is given in the following table.

	Job			
Person	**1**	**2**	**3**	**4**
A	1	8	15	22
B	13	18	23	28
C	13	18	23	28
D	19	23	27	31

Assign one person to one counter to minimize the total cost.

Solution:

After applying steps 1 to 3 of the Hungarian Method, we obtain the following matrix.

Table

	Job			
Person	**1**	**2**	**3**	**4**
A	$\boxed{0}$	3	6	9
B	$\cancel{0}$	1	2	3
C	$\cancel{0}$	1	2	3
D	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Now by applying the usual procedure, we get the following matrix.

Table

Job				
Person	**1**	**2**	**3**	**4**
A	$\boxed{0}$	2	5	8
B	~~0~~	$\boxed{0}$	1	2
C	~~0~~	~~0~~	1	2
D	1	~~0~~	$\boxed{0}$	~~0~~

The resulting matrix suggest the alternative optimal solutions as shown in the following tables.

Table

Job				
Person	**1**	**2**	**3**	**4**
A	$\boxed{0}$	2	4	7
B	~~0~~	$\boxed{0}$	~~0~~	1
C	~~0~~	~~0~~	$\boxed{0}$	1
D	2	1	~~0~~	$\boxed{0}$

Job				
Person	**1**	**2**	**3**	**4**
A	$\boxed{0}$	2	4	7
B	~~0~~	~~0~~	$\boxed{0}$	1
C	~~0~~	$\boxed{0}$	~~0~~	1
D	2	1	~~0~~	$\boxed{0}$

The persons B & C may be assigned either to job 2 or 3.
The two alternative assignments are:

A1 + B2 + C3 + D4
1 + 18 + 23 + 31 = 73

A1 + B3 + C2+ D4
1 + 23 + 18 + 31 = 73

An Application - Airline Crew Assignment

The Hungarian method discussed in the previous sections can also be utilized to plan the assignment of crew members at different locations by a transport company. To further enhance your understanding about assignment models, we provide the following examples.

Example

Best-ride airlines that operates seven days a week has the following time-table.

Flight No.	Delhi - Mumbai	
	Departure	Arrival
1	7.00 AM	8.00 AM
2	8.00 AM	9.00 AM
3	1.00 PM	2.00 PM
4	6.00 PM	7.00 PM

Flight No.	Mumbai-Delhi	
	Departure	Arrival
101	8.00 AM	9.00 AM
102	9.00 AM	10.00 AM
103	12.00 Noon	1.00 PM
104	5.00 PM	6.00 PM

Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover. For each pair also mention the city where crew should be based.

Solution

To determine optimal assignments, first we calculate layover times from the above time table.

Calculating values for table 1 (layover time)

First Row

First cell

Arrival time (Mumbai) = 8.00 AM & Departure time (Mumbai) = 8.00 AM
Difference between arrival and departure = 24 hours (layover time)

Second cell

Arrival time (Mumbai) = 8.00 AM & Departure time (Mumbai) = 9.00 AM
Difference between arrival and departure = 25 hours (layover time)

Third cell

Arrival time (Mumbai) = 8.00 AM & Departure time (Mumbai) = 12.00 Noon
Difference between arrival and departure = 28 hours (layover time)

Fourth cell

Arrival time (Mumbai) = 8.00 AM & Departure time (Mumbai) = 5.00 PM
Difference between arrival and departure = 9 hours (layover time)

Similarly, values for other rows can be calculated.

Table 1

Crew based at Delhi				
Flight No.	**101**	**102**	**103**	**104**
1	24	25	28	9
2	23	24	27	8
3	18	19	22	27
4	13	14	17	22

Calculating values for table 2 (layover time)

First Column

First cell

Arrival time (Delhi) = 9.00 AM & Departure time (Delhi) = 7.00 AM
Difference = 22 hours

Second cell

Arrival time (Delhi) = 9.00 AM & Departure time (Delhi) = 8.00 AM
Difference = 23 hours

Third cell

Arrival time (Delhi) = 9.00 AM & Departure time (Delhi) = 1.00 PM
Difference = 28 hours

Fourth cell

Arrival time (Delhi) = 9.00 AM & Departure time (Delhi) = 6.00 PM
Difference = 9 hours

Similarly, values for other columns can be calculated.

Table 2

Crew based at Mumbai				
Flight No.	**101**	**102**	**103**	**104**
1	22	21	18	13
2	23	22	19	14
3	28	27	24	19
4	9	8	5	24

The composite layover time matrix (table 3) is obtained by selecting the smaller element from the two corresponding elements of table 1 & 2. The layover time marked with **(*)** represents that the crew is based at Mumbai, otherwise based at Delhi. For example, corresponding to flight no.1 and 101 in table 1 & 2, we select the minimum between (24, 22), i.e., 22 for Mumbai. Therefore, this element is marked with **(*)**. In this way, table 3 is completed and shown below.

Table 3

Flight No.	101	102	103	104
1	22*	21*	18*	9
2	23	22*	19*	8
3	18	19	22	19*
4	9*	8*	5*	22

Now the above problem can be easily solved by Hungarian method. The following matrix shows the assignments.

Table 4

Flight No.	101	102	103	104
1	13*	11*	9*	$\boxed{0}$
2	15	13*	11*	$\cancel{0}$
3	$\boxed{0}$	$\cancel{0}$	4	1*
4	4*	2*	$\boxed{0}$*	17

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix.

Table 5

Flight No.	101	102	103	104
1	13*	11*	9*	0
2	15	13*	11*	~~0~~
3	0	~~0~~	4	1*
4	4*	2*	0*	17

Select the smallest element from all the uncovered elements, i.e., 9. Subtract this element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Repeating step 3 of the Hungarian algorithm, we obtain a solution which is shown in the following table.

Table 6

Flight No.	101	102	103	104
1	4*	2*	0*	~~0~~
2	6	4*	2*	0
3	0	~~0~~	4	10*
4	4*	2*	~~0~~*	26

Repeating the same procedure, we get the following matrix.

Table 7

Flight No.	101	102	103	104
1	2*	0*	~~0~~*	~~0~~
2	4	2*	2*	0
3	0	~~0~~	6	12*
4	2*	~~0~~*	0*	26

The optimal solution is 21 + 8 + 18 + 5 = 52 hours.

Example - 2

Universal bus service operates seven days in a week. A trip from Delhi to Rajpura takes six hours by bus. A typical time table of the bus service in both directions is given below:

Delhi - Rajpura		
Bus No.	**Departure from Delhi**	**Arrival at Rajpura**
A	6.00	12.00
B	7.30	13.30
C	11.30	17.30
D	19.00	1.00
E	00.30	6.30

Rajpura - Delhi		
Bus No.	**Departure from Rajpura**	**Arrival at Delhi**
1	5.30	11.30
2	9.00	15.00
3	15.00	21.00
4	18.30	00.30
5	00.00	6.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service times. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before the return trip and should not wait for more than 24 hours for the return trip. The company has residential facilities for the crew of Delhi as well as of Rajpura. Find which line of service be connected with which other line so as to reduce the waiting time to the minimum.

Solution:

If bus no. A is combined with bus no. 1, the crew after arriving at Rajpura at 12 noon starts at 5.30 next morning. Thus, the waiting time is 17.30 hours as shown in **table 1**. Some of the assignments are infeasible, e.g., bus no. 3 leaves Rajpura at 15.00 hours. Thus, the crew of bus no. A after reaching Rajpura at 12 noon are unable to take the minimum required rest of four hours, if they are asked to leave by bus no. 3. Hence, A3 is an infeasible assignment. Thus, its cost is M, a large positive number.

Table 1

Crew based at Delhi					
Bus no.	**1**	**2**	**3**	**4**	**5**
A	17.30	21	M	6.30	12
B	16	19.30	M	5	10.30
C	12	15.30	21.30	M	6.30
D	4.30	8	14	17.30	23
E	23	M	8.30	12	17.30

Similarly, if the crew are assumed to stay at Rajpura, then the waiting times of the crew in hours at Delhi are given in **table 2**.

Table 2

Crew based at Rajpura					
Bus no.	**1**	**2**	**3**	**4**	**5**
A	18.30	15	9	5.30	24
B	20	16.30	10.30	7	M
C	24	20.30	14.30	11	5.30
D	7.30	M	22	18.30	13
E	13	9.30	M	24	18.30

The composite layover time matrix (table 3) is obtained by selecting the smaller element from the two corresponding elements of table 1 & 2. The layover time marked with **(*)** represents that the crew is based at Delhi, otherwise based at Rajpura.

Table 3

Bus no.	1	2	3	4	5
A	17.30*	15	9	5.30	12*
B	16*	16.30	10.30	5*	10.30*
C	12*	15.30*	14.30	11	5.30
D	4.30*	8*	14*	17.30*	13
E	13	9.30	8.30*	12*	17.30*

Table 4

To simplify the problem, we multiply every element of the matrix with 2. Thus, the resulting matrix is:

Bus no.	1	2	3	4	5
A	35*	30	18	11	24*
B	32*	33	21	10*	21*
C	24*	31*	29	22	11
D	9*	16*	28*	35*	26
E	26	19	17*	24*	35*

Now solve the above problem by the Hungarian method.

The optimal solution is 4.30 + 9.30 + 9 + 5 + 5.30 = 33.30 hours

Travelling Salesman Problem

This humorously named problem refers to the following situation:
A traveling salesman, named Rover plans to visit each of n cities. He wishes to visit each city once and only once, arriving back to city from where he started. The distance between City i and City j is c_{ij}. What is the shortest tour Rover can take?

If there are n cities, there are (n - 1)! Possible ways for his tour. For example, if the number of cities to be visited is 5, then there are 4! different combinations. Such type of problems can be solved by the assignment method.
Let c_{ij} be the distance (or cost or time) between City i to City j and

$$x_{ij} = \begin{cases} 1 \text{ if a tour includes travelling from city i to city j (for } i \neq j) \\ 0 \text{ otherwise} \end{cases}$$

The following example will help you in understanding the travelling salesman problem.

Example

A traveling salesman, named Rolling Stone plans to visit five cities 1, 2, 3, 4 & 5. The travel time (in hours) between these cities is shown below:

	To				
From	**1**	**2**	**3**	**4**	**5**
1	∞	5	8	4	5
2	5	∞	7	4	5
3	8	7	∞	8	6
4	4	4	8	∞	8
5	5	5	6	8	∞

How should Mr. Rolling Stone schedule his touring plan in order to **minimize** the total travel **time**, if he visits each city once a week?

Solution

After applying steps 1 to 3 of the Hungarian method, we get the following assignments.

Table

	To				
From	**1**	**2**	**3**	**4**	**5**
1	∝	1	3	$\boxed{0}$	1
2	1	∝	2	$\cancel{0}$	1
3	2	1	∝	2	$\boxed{0}$
4	$\boxed{0}$	$\cancel{0}$	3	∝	4
5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3	∝

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix.

Table

	To				
From	**1**	**2**	**3**	**4**	**5**
1	∝	1	3	$\boxed{0}$	1
2	1	∝	2	$\cancel{0}$	1
3	2	1	∝	2	$\boxed{0}$
4	$\boxed{0}$	$\cancel{0}$	3	∝	4
5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3	∝

Select the smallest element from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment. Repeating step 3 on the reduced matrix, we get the following assignments.

Table

	To				
From	**1**	**2**	**3**	**4**	**5**
1	∝	$\cancel{0}$	2	$\boxed{0}$	1
2	$\boxed{0}$	∝	1	$\cancel{0}$	1
3	1	$\cancel{0}$	∝	2	$\boxed{0}$
4	$\cancel{0}$	$\boxed{0}$	3	∝	5
5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∝

The above solution suggests that the salesman should go from city 1 to city 4, city 4 to city 2, and then city 2 to 1 (original starting point). The above solution is not a solution to the traveling salesman problem as he visits city 1 twice.

The next best solution can be obtained by bringing the minimum non-zero element, i.e., 1 into the solution. Please note that the value 1 occurs at four places. We will consider all the cases separately until the acceptable solution is obtained. To make the assignment in the cell (2, 3), delete the row & the column containing this cell so that no other assignment can be made in the second row and third column.

Now, make the assignments in the usual manner as shown in the following table.

Table

	To				
From	1	2	3	4	5
1	∞	~~0~~	2	$\boxed{0}$	1
2	~~0~~	∞	$\boxed{1}$	~~0~~	1
3	1	~~0~~	∞	2	$\boxed{0}$
4	~~0~~	$\boxed{0}$	3	∞	5
5	$\boxed{0}$	~~0~~	~~0~~	4	∞

He starts from city 1 and goes to city 4; from city 4 to city 2; from city 2 to city 3; from city 3 to city 5; from city 5 to city 1.

Substituting values from original table:
4 + 7 + 6+ 4 + 5 = 26 hours.

Self Test Questions

Theory

1. Give the mathematical formulation of an assignment problem.

2. Describe the Hungarian Method of solving the assignment problem.

3. Explain the difference between a transportation and an assignment problem.

4. Explain Unbalanced Assignment Problem with the help of an example.

5. Explain Maximization In An Assignment Problem with the help of an example.

Practical

1. In a textile sales emporium, four salesmen A, B, C and D are available to four counters W, X, Y and Z. Each salesman can handle any counter. The service in (hour) of each counter when manned by each salesman is given below:

Salesman				
Counter	**A**	**B**	**C**	**D**
W	41	32	39	52
X	22	29	49	65
Y	27	39	60	51
Z	45	50	48	52

How should the salesman be allocated appropriate counters so as to minimize the service time?

2. Five wagons are available at five stations 1, 2, 3, 4 and 5. These are required at five stations I, II, III, IV and V. The mileages between various stations are given by:

Station					
Wagon	**I**	**II**	**III**	**IV**	**V**
1	10	5	9	18	11
2	13	19	6	12	14
3	3	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

3. A department head has four subordinates and four tasks. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of the time each man would take to perform each task is given below. How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Man				
Job	**A**	**B**	**C**	**D**
W	8	26	17	11
X	13	28	4	26
Y	38	19	18	15
Z	19	26	24	10

4. Four different buildings W, X, Y and Z are to be constructed in a Maternity Hospital by four different contractors A, B, C and D. It is also assumed that one contractor is

required to build one building. Each contractor has submitted bids for the four buildings. The bids have been shown below. The problem is to determine which building is to be awarded to each contractor to keep the cost of construction of the four buildings at a minimum.

	Contractor			
Building	**A**	**B**	**C**	**D**
W	48	48	50	44
X	56	60	60	68
Y	96	94	90	85
Z	42	44	54	46

5. Solve the following assignment problems:

(a)

	Man			
Job	**I**	**II**	**III**	**IV**
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

(b)

	Man				
Job	**I**	**II**	**III**	**IV**	**V**
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

(c)

	Man				
Job	**I**	**II**	**III**	**IV**	**V**
1	1	3	2	3	6
2	2	4	3	1	5
3	5	6	3	4	6
4	3	1	4	2	2
5	1	5	6	5	4

(d)

Man				
Job	**I**	**II**	**III**	**IV**
1	12	30	21	15
2	18	33	9	31
3	44	25	24	21
4	23	30	28	14

(e)

Man				
Job	**I**	**II**	**III**	**IV**
1	2	3	4	5
2	4	5	6	7
3	7	8	9	8
4	3	5	8	4

(f)

Man					
Job	**I**	**II**	**III**	**IV**	**V**
1	12	8	7	15	4
2	7	9	1	14	10
3	9	6	12	6	7
4	7	6	14	6	10
5	9	6	10	10	6

6. One car is available at each of the station 1, 2, 3, 4, 5, 6 and one car is required at each of the stations 7, 8, 9, 10, 11, 12. The distances between the various stations are given in the matrix below. How should the cars be dispatched so as to minimize the total mileage covered ?

	7	**8**	**9**	**10**	**11**	**12**
1	41	72	39	52	25	51
2	22	29	49	65	81	50
3	27	39	60	51	32	32
4	45	50	48	52	65	43
5	29	40	39	26	30	33
7	82	40	40	60	51	30

7. The Manager of a company wants to assign X, Y and Z to regional offices Delhi, Mumbai, Kolkata and Chennai. The cost of reallocation (in Rupees) of the three officers at the four regional offices are given below:

Office				
Officer	**Delhi**	**Mumbai**	**Kolkata**	**Chennai**
X	16000	22000	24000	20000
Y	10000	32000	26000	16000
Z	10000	20000	46000	30000

How should the officers be allocated appropriate regional offices so as to minimize the cost?

8. A company has four sales territories and four salesmen available for assignment. The sales territories are not equally rich in their sales potential and the salesmen also differ in their selling ability. The following matrix gives the sales (in thousand rupees) for each salesman to be assigned to each territory. How should the territories be assigned to salesmen so as to maximize the total sales?

Territory				
Salesman	**A**	**B**	**C**	**D**
W	42	35	28	21
X	30	25	20	15
Y	30	25	20	15
Z	24	20	16	12

9. One-Ride Airlines that operates seven days a week has the time-table given below. Crews must have a minimum layover of 6 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in smaller layover.

Delhi - Jaipur		
Flight No.	**Departure**	**Arrival**
101	7.00 AM	8.15 AM
102	8.30 AM	9.45 AM
103	1.00 PM	2.15 PM
104	6.00 PM	7.15 PM

Jaipur - Delhi		
Flight No.	**Departure**	**Arrival**
201	8.00 AM	9.15 AM
202	9.00 AM	10.15 AM
203	12.00 Noon	1.15 PM
204	5.45 PM	7.00 PM

For each pair also mention the city where crew should be based.

10. Railways that operates seven days a week has the time-table given below. Crews must have a minimum layover of 2 hours between trains. Obtain the pairing of trains that minimizes layover time away from home. For any given pairing, the crew will be based at the city that results in smaller layover.

Delhi - Karnal		
Train No.	**Departure**	**Arrival**
1010	7.00 AM	9.00 AM
1020	8.30 AM	10.30 AM
1030	1.00 PM	2.00 PM
1040	6.50 PM	8.50 PM

Karnal - Delhi		
Train No.	**Departure**	**Arrival**
2010	8.00 AM	10.00 AM
2020	9.00 AM	11.00 AM
2030	12.20 PM	2.20 PM
2040	5.45 PM	7.45 PM

For each pair also mention the city where crew should be based.

11. One ride airline operating 7 days a week has given the following timetable. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any pairing the crew will be based at the city that results in the smaller layover.

Chennai - Mumbai		
Flight No.	**Departure**	**Arrive**
A1	6 AM	8 AM
A2	8 AM	10 AM
A3	2 PM	4 PM
A4	8 PM	10 PM

Mumbai - Chennai		
Flight No.	**Departure**	**Arrive**
B1	8 AM	10 AM
B2	9 AM	11 AM
B3	2 PM	4 PM
B4	7 PM	9 PM

Chapter – 7

Integer Programming

Introduction

The general linear programming model depends on the assumption of divisibility. In other words, the decision variables are allowed to take non-negative integer as well as fractional values. However, we quite often face situations where the planning models contain integer valued variables. For instance, trucks in a fleet, generators in a powerhouse, pieces of equipment, investment alternatives and there are a myriad of other examples. In all such cases, an integer solution is desired, which can be easily obtained by rounding off the fractional values of the variables. But, rounding-off may result in sub-optimal or infeasible solutions. To overcome such difficulties, a different optimization model, which is referred to as integer programming has been developed.

Definition

Integer (or discrete) programming problem is a type of problem in which some, or all, of the variables are allowed to take only integral values.

The focus of this chapter is on solution techniques for integer programming models.

The mathematical model of the problem is as follows:

Optimize (maximize or minimize) $\sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq) \ b_i; \ i = 1, 2, \ldots.., m$

$x_j \geq 0; \ j = 1, 2, \ldots.., n$

x_j **integer valued**; $j = 1, 2, \ldots.., p \leq n$

If all the variables are restricted to take only integral values (i.e., p = n), the model is called a **pure** integer programming problem. To the contrary, if some variables are restricted to take only integer values, and the remaining are free to take any non-negative values, then it is called a **mixed** integer programming problem. When the decision variables are required to take value of either zero or one, it is called **zero-one** programming.

The mathematical model of **zero-one** programming is as follows:

Optimize (maximize or minimize) $\sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq) \ b_i; \ i = 1, 2, \ldots.., m$

$x_j = 0$ or 1; $j = 1, 2, \ldots.., n$

Model Formulation

The effective use and applications require, as a first step, the formulation of the model when the problem is presented.

In this section, we illustrate the formulation of integer programming problems. The best way to explain a topic is through examples. So consider the following examples.

Example 1

You have entered in a treasure cave (with a password KHUL JA SIM SIM) full of three types of valuable stones, amethyst (A), ruby (R), and topaz (T). Each piece of A, R, and T weighs 3, 2, 2 kg., and is known to have a value of 4, 3, 1 crore respectively. You have got a bag that can carry a maximum of 11 kg. Your problem is to decide on how many pieces of each type to carry, within the capacity of your bag, so as to **maximize the total value carried**. **The stones cannot be broken.**

Solution:

We start the formulation exercise by defining the decision variables.

Let
x_1 = Number of amethysts to be carried
x_2 = Number of rubies to be carried
x_3 = Number of topaz to be carried

The objective function here is to **maximize** the total value carried, which is given by the linear function.

Maximize $4x_1 + 3x_2 + x_3$

Since one amethyst is of 3 kg, one ruby is of 2 kg, one topaz is of 2 kg, and the capacity of the bag being 11 kg., the constraint can be expressed as

$3x_1 + 2x_2 + 2x_3 \leq 11$

Finally, we note the fact that stones cannot be broken, i.e., the variables have to take discrete values, which may be stated algebraically as

x_1, x_2 and x_3 are all non-negative integers.

Thus, we have the following formulation:

Maximize $(4x_1 + 3x_2 + x_3)$

subject to

$3x_1 + 2x_2 + 2x_3 \leq 11$
x_1, x_2 and x_3 are all non-negative integers.

Go-No-Go Decisions

Example 2

www.universalteacher.com wants to take up four projects. However, because of budget limitations, not all the projects can be selected. It is known that project j has a present value of c_j and would require an investment of a_{jt} in period t. The capital available in period t is b_t. The problem is to choose projects so as to maximize present value, without violating the budget constraints. Formulate the problem as an I.P.

Solution:

For choice situations of 'yes-no', 'go-no-go' type, where the objective is to determine whether or not a particular activity is to be undertaken, integer binary variables that can take a value of 0 or 1, can be used to represent the decision variables. Here, we find that for each project, we want to find out whether it should be taken up or not, as such we define four decision variables x_j (j = 1, 2, 3, 4) corresponding to each project, and

$$x_j = \begin{cases} 1 \text{ if a if project is selected.} \\ 0 \text{ otherwise} \end{cases}$$

Then, the objective function and the constraints can be expressed in terms of the decision variables, to give the required formulation:

Maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{jt} x_j \leq b_t$, for all period t.

$x_j = 1$ or 0, for all projects.

Basically, there are two algorithms to determine the optimal solution for an integer programming problem. One of these is the cutting plane algorithm devised by Gomory and the other is the branch & bound algorithm developed by Land & Doig. The next section concentrates on the cutting plane method.

Cutting Plane Algorithm

Historically, the first method for solving I.P.P. was the cutting plane method. This method is for the pure integer programming model. The procedure is, first, ignore the integer stipulations, and solve the problem as an ordinary LPP. If the solution satisfies the integer restrictions, then an optimal solution for the original problem is found. Otherwise, at each iteration, additional constraints are added to the original problem. These constraints are added to reduce or cut the solution space in every successive iteration, ruling out the current fractional solution, while ensuring that no integer solution is excluded in the process. The method terminates as soon as an integer-valued is obtained.

To summarize the approach, a series of steps are stated below.

Steps

1. Use the simplex method to find an optimal solution of the problem, ignoring the integer condition.
2. Examine the optimal solution. Terminate the iterations if all the basic variables have integer values. Otherwise, construct a Gomory's fractional cut from the row, which has the largest fractional part, and add it to the original set of constraints.
   Gomory's constraint

$- \Sigma f_i x_j \leq -f_l$
$- \Sigma f_i x_j + S_l = -f_l$
Where:
f_l - fractional part
S_l - slack variable

In case of a tie, **you are at liberty to choose any one arbitrarily**.

3. Now, find a new linear programming solution. If the solution thus obtained is integral valued, then this is the required optimal solution of the original I.P.P.; otherwise, return to step 2.

Graphical Method

This section deals with the geometric representation of an integer programming problem. To illustrate the concept of cutting plane method through graphical method, consider again the following problem.

Example

Maximize $z = x_1 + 4x_2$

Subject to
$2x_1 + 4x_2 \leq 7$
$5x_1 + 3x_2 \leq 15$

x_1, x_2 are integers ≥ 0

Solution:

First, solve the above problem by applying the **simplex method**.

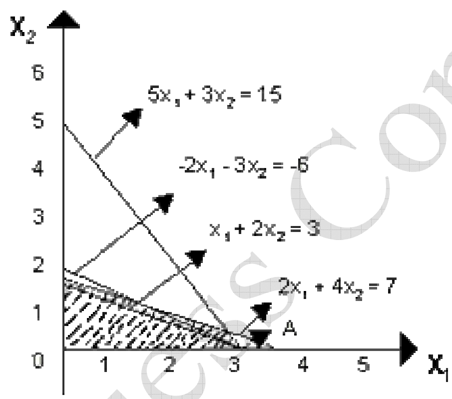
After introducing slack variables, we have
$2x_1 + 4x_2 + x_3 = 7$
or $x_3 = 7 - 2x_1 - 4x_2$ ....(i)
$5x_1 + 3x_2 + x_4 = 15$
or $x_4 = 15 - 5x_1 - 3x_2$ ....(ii)

Gomory's constraint
$- (1/2x_1 - 3/4\ x_3) \leq -3/4$ ...(iii)

Substituting the value of x_3 in equation (iii).
$- 1/2x_1 + 3/4 (7 -2x_1 - 4x_2) \leq -3/4$
$-2x_1 - 3x_2 \leq -6$ ....(iv)

Gomory's constraint
$-1/2x_3 \leq -1/2$ ....(v)

Substituting the value of x_3 in equation (v)
$-1/2 (7 - 2x_1 - 4x_2) \leq -1/2$
$x_1 + 2x_2 \leq 3$ ....(vi)



The inclusion of two cuts($-2x_1 - 3x_2 \leq -6$, $x_1 + 2x_2 \leq 3$) give the new corner point A where $x_1 = 3$, $x_2 = 0$ and $z = 3$

Branch & Bound Method

The branch & bound method can be used to solve problems containing a few integer valued variables. It can be applied to both **mixed** & **pure** integer programming problems. This method partitions the area of feasible solution into smaller parts until an optimal solution is obtained.

Steps

1. First, solve the given problem as an ordinary LPP.
2. Examine the optimal solution. Terminate the iterations if the optimal solution to the LPP satisfies the integer constraints. Otherwise, go to step 3.
3. Divide the problem into two parts.
   Problem 1: xk ≤ [t]

max. z = cx
subject to
ax ≤ b
xk ≤ [t]
x ≥ 0

Problem 2: xk ≥ [t] + 1
max. z = cx
subject to
ax ≤ b
xk ≥ [t] + 1
x ≥ 0

where [t] is the largest integer.

4. Now, solve problem 1 & 2 separately.
5. If for any of the sub-problems, optimal integer solution is obtained, then that problem is not further branched. Otherwise, move to step 3.

Self Test Questions

Theory

1. What do you understand by integer programming problem?

2. Discuss the need of integer programming.

Practical

1. Maximize $z = 7x_1 + 9x_2$

subject to
$-x_1 + 3x_2 \leq 6$
$7x_1 + x_2 \leq 35$

where x_1, x_2 are integers ≥ 0

2. Maximize $z = x_1 + 2x_2$

subject to
$2x_2 \leq 7$
$x_1 + x_2 \leq 7$
$2x_1 \leq 11$

Where x_1, x_2 are integers ≥ 0

3. Maximize $z = 2x_1 + 6x_2$

Subject to
$3x_1 + x_2 \leq 5$
$4x_1 + 4x_2 \leq 9$

Where x_1, x_2 are integers ≥ 0

4. Solve the following by Branch & Bound method

Maximize $z = 5x_1 + 9x_2$

Subject to
$-x_1 + 5x_2 \leq 3$
$5x_1 + 3x_2 \leq 27$

Where x_1, x_2 are non negative integers

Chapter – 8

Goal Programming

Introduction

According to the classical economic theory, profit maximization is one of the most widely accepted **goal** of management. However, in today's dynamic business environment, there is no single universal objective for all organizations. Organizational objectives vary according to the characteristics, types, environmental conditions, etc. of the organization. Various goals may be expressed in different units of measurement such as rupees, hours, tons, etc. Often multiple goals of management are in conflict or are achievable only at the expense of other goals. Consequently, one of the most important and difficult problem is to achieve an equilibrium between these conflicting objectives.

What is goal programming?

Goal programming is a powerful technique that is capable of handling multiple decision criteria. In other words, goal programming is a powerful tool to tackle multiple and incompatible goals of an enterprise.

Linear Programming versus Goal Programming

Unquestionably, linear programming models are among the most commercially successful applications of operations research. But, one of the limitations of linear programming is that its objective function is unidimensional, i.e., the decision maker strives for a single objective, such as profit maximization or cost minimization. To the contrary, in goal programming, the objective function contains primarily the deviational variables that represent each goal or sub-goal.

Another limitation of LP is that the management must accurately quantify the relationship of the variables in cardinal values (numbers that express exact values such as 1, 3, 4.5, etc.). But, when the goals are incommensurable, then these goals cannot be assigned cardinal values. These two shortcomings can be overcome by using the goal programming technique.

Model Formulation

It is important to consider model formulation before launching into the details of goal programming solutions. Model formulation is the process of transforming a real word decision problem into an operations research model. A key to successful application of goal programming is the **ability** to recognize when a problem can be solved by goal programming and to formulate the corresponding model.

The approach to formulate the goal programming model is similar to that of linear programming model. The mathematical model is given as follows:

Minimize $\sum_{i=1}^{m} w_i (d_i^- + d_i^+)$

subject to

$\sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i$; i = 1, 2, ..., m

and $x_j, d_i^-, d_i^+ \geq 0$ for all i, j

The objective function contains primarily the deviational variables (d_i^- & d_i^+) that represent each goal or subgoal.

The deviational variables are represented as both positive and negative deviations from each goal or subgoal. Thus, the objective function becomes the minimization of these deviations based on the relative importance or priority assigned to them.

The steps in model formulation can be summarized as follows:

Steps

1. **Define Variables and Constants.** The first step in model formulation is the definition of decision variables (x_1, x_2, ..., x_n) and the right hand side constants. The right hand side constants may be either available resources or specified goal levels.
2. **Formulate Constraints.** The next step is to formulate a set of constraints. A constraint may be either a system constraint or a goal constraint.
3. **Develop the Objective Function.** Through the analysis of the decision maker's goal structure, the objective function must be developed. **If goals are classified in k ranks, the preemptive priority factors (symbolized by P1, P2, and so on) should be assigned to deviational variables according to their order of importance.** If necessary, differential weights must be assigned to deviational variables at the same priority level.

Model Formulation

Example

The Japan Life Company produces two products- A and B. According to the past experience, production of either product A or product B requires an average of one hour in the plant. The plant has a normal production capacity of 300 hours a month. The marketing department of the firm reports that because of limited market, the maximum number of product A and product B that can be sold in a month are 140 and 200 respectively. The net profit from the sale of product A and product B are Rs. 600 and Rs. 200 respectively. The manager has set the following goals.

P_1: The first goal is to avoid any underutilization of normal production capacity.
P_2: He wants to sell maximum possible units of product A and B. Since the net profit from the sale of product A is thrice the amount from Product B, therefore, the manager has thrice as much desire to achieve sales for product A as for Product B.
P_3: He wants to minimize the overtime operation of the plant as much as possible.

Solution:

Production capacity

Let x_1 = number of units of product A
x_2 = number of units of product B

Since overtime operation of the plant is allowed to a certain extent, the constraint can be written as
$x_1 + x_2 + d_1^- - d_1^+ = 300$

Where
d_1^- = underutilization (idle) of production capacity and
d_1+ = overtime operation of the normal production capacity.

Sales constraints

In this case, the maximum sales for product A and product B are set at 140 and 200 respectively. Hence, it is assumed that overachievement of sales beyond the maximum limits is impossible. Then, the sales (market) constraints can be expressed as

$x_1 + d_2^- = 140$
$x_2 + d_3^- = 200$

Where d_2^- and d_3^- are the underachievement of the sales goal for product A and B respectively.

Therefore, the complete goal programming model can be written as

Minimize z = $P_1d_1^- + 3P_2d_2^- + P_2d_3^- + P_3d_1^+$

Subject to

$x_1 + x_2 + d_1^- - d_1^+ = 300$
$x_1 + d_2^- = 140$
$x_2 + d_3^- = 200$

And $x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0$

Graphical Method

This section deals with geometric representation of a goal programming problem. The graphical method of solving goal programming problem is quite similar to the graphical method of linear programming.

Example

Minimize z = $P_1d_1^- + 2P_2d_2^- + P_2d_3^- + P_3d_1^+$

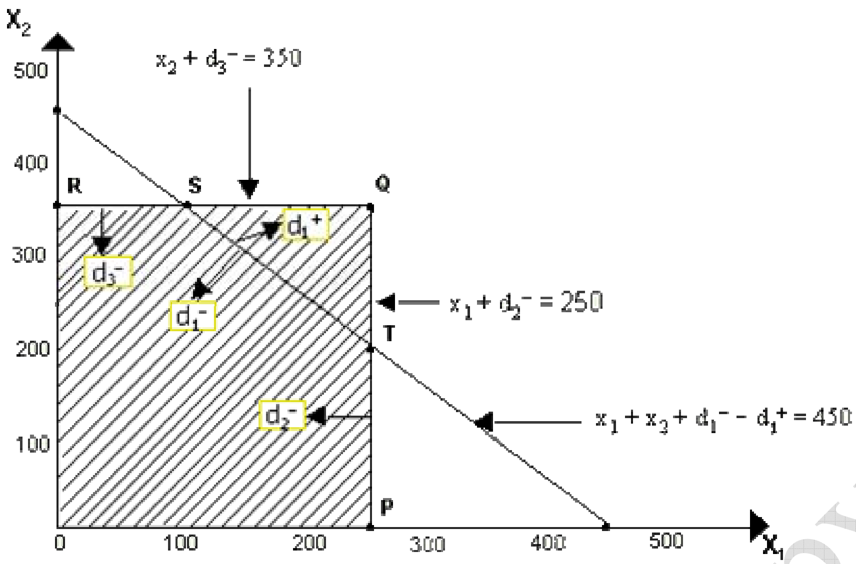
subject to

$x_1 + x_2 + d_1^- - d_1^+ = 450$
$x_1 + d_2^- = 250$
$x_2 + d_3^- = 350$

$x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0$

Solution:

The problem is graphed in the following figure.



The region bounded by OPQR represents the solution set.

The optimal solution to the problem is given below:
$x_1 = 250$, $x_2 = 350$, $d_1^- = 0$, $d_2^- = 0$, $d_3^- = 0$, $d_1^+ = 150$

Modified Simplex Method of Goal Programming

We think that the best way to explain the simplex method of goal programming is through an example. Here, we proceed directly to the consideration of a numerical example.

Example 1

Minimize z = $P_1d_1^- + P_2d_4^+ + 5P_3d_2^- + 3P_3d_3^- + P_4d_1^+$

subject to

$x_1 + x_2 + d_1^- - d_1^+ = 80$
$x_1 + d_2^- = 70$
$x_2 + d_3^- = 45$
$d_1^+ + d_4^- - d_4^+ = 10$

$x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+, d_4^-, d_4^+ \geq 0$

Solution.

Substituting $x_1 = 0$, $x_2 = 0$, $d_1^+ = 0$ & $d_4^+ = 0$

Therefore, $d_1^- = 80$, $d_2^- = 70$, $d_3^- = 45$, $d_4^- = 10$

The first four rows of **table 1** are set up in the same way as for the Simplex Method. The next four rows stand for priority goal levels. The goal levels P_1, P_2, P_3 and P_4 are arranged in descending order.

Table 1

	c_j	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3	d_4^-	d_1^+	d_4^+	Solution values b ($=X_B$)
P_1	d_1^-	1	1	1	0	0	0	-1	0	80
$5P_3$	d_2^-	1	0	0	1	0	0	0	0	70
$3P_3$	d_3^-	0	1	0	0	1	0	0	0	45
0	d_4^-	0	0	0	0	0	1	1	-1	10
z_j-c_j	P_4	0	0	0	0	0	0	-1	0	0
	P_3	5	3	0	0	0	0	0	0	485
	P_2	0	0	0	0	0	0	0	-1	0
	P_1	1	1	0	0	0	0	-1	0	80

Calculating values for the index row ($z_j - c_j$)

z_j - c_j = (Elements in cB Column) X (corresponding elements in x_j columns) – c_j (Priority factors assigned to deviational variables)

Column x_1
$z_1 - c_1 = P_1 \text{ X } 1 + 5P_3 \text{ X } 1 + 3P_3 \text{ X } 0 + 0 \text{ X } 0 – 0 = P_1 + 5P_3$

Column x_2
$z_2 - c_2 = P_1 \text{ X } 1 + 5P_3 \text{ X } 0 + 3P_3 \text{ X } 1 + 0 \text{ X } 0 – 0 = P_1 + 3P_3$

Column d_1^-
$z_3 - c_3 = P_1 X 1 + 5P_3 X 0 + 3P_3 X 0 + 0 X 0 – P_1 = 0$

Column d_2^-
$z_4 - c_4 = P_1 X 0 + 5P_3 X 1 + 3P_3 X 0 + 0 X 0 – 5P_3 = 0$

Column d_3^-
$z_5 - c_5 = P_1 X 0 + 5P_3 X 0 + 3P_3 X 1 + 0 X 0 – 3P_3 = 0$

Column d_4^-
$z_6 - c_6 = P_1 X 0 + 5P_3 X 0 + 3P_3 X 0 + 0 X 1 – 0 = 0$

Column d_1^+
$z_7 - c_7 = P_1 X (-1) + 5P_3 X 0 + 3P_3 X 0 + 0 X 1 – P_4 = –P_1 – P_4$

Column d_4^+
$z_8 - c_8 = P_1 X 0 + 5P_3 X 0 + 3P_3 X 0 + 0 X (-1) – P_2 = – P_2$

Column X_B
$z_B - c_B = P_1 X 80 + 5P_3 X 70 + 3P_3 X 45 + 0 X 10 = 80P_1 + 485P_3$

Since, P_1, P_2, P_3 and P_4 are not commensurable, we list their coefficients separately in their rows in the simplex criterion (z_j - c_j) as shown in **table 1**.

Key column

The key column can be determined by selecting the largest positive element in z_j - c_j row at the highest priority goal level. In table 1, the largest positive element 1 in the P_1 row occurs at two places. In order to break this tie, check the next lower priority goal levels. Since the largest positive element is 5 in P_3 row, therefore, column under x_1 becomes the key column.

Key row

Minimum positive value = (80/1, 70/1) = 70
So d_2^- row is the key row.

Pivot element = 1
Therefore, d_2^- departs and x_1 enters.

Table 2

	c_j	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
c_B	**Basic variables B**	x_1	x_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	**Solution values b (=X_B)**
P_1	d_1^-	0	1	1	-1	0	0	-1	0	10
0	x_1	1	0	0	1	0	0	0	0	70
$3P_3$	d_3^-	0	1	0	0	1	0	0	0	45
0	d_4^-	0	0	0	0	0	1	1	-1	10
z_j-c_j	P_4	0	0	0	0	0	0	-1	0	0
	P_3	0	3	0	-5	0	0	0	0	135
	P_2	0	0	0	0	0	0	0	-1	0
	P_1	0	1	0	-1	0	0	-1	0	10

Key column = x_2 column
Minimum positive value = Min(10/1, 45/1) = 10
So d_1^- row is the key row.
d_1^- departs & x_2 enters

Table 3

	c_j	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
c_B	**Basic variables B**	x_1	x_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	**Solution values b (=X_B)**
0	x_2	0	1	1	-1	0	0	-1	0	10
0	x_1	1	0	0	1	0	0	0	0	70
$3P_3$	d_3^-	0	0	-1	1	1	0	1	0	35
0	d_4^-	0	0	0	0	0	1	1	-1	10
z_j-c_j	P_4	0	0	0	0	0	0	-1	0	0
	P_3	0	0	-3	-2	0	0	3	0	105
	P_2	0	0	0	0	0	0	0	-1	0
	P_1	0	0	-1	0	0	0	0	0	10

Key column = d_1+ column
Minimum positive value = Min(35/1, 10/1) = 10
d_4^- departs & d_1+ enters

Table 4

	c_j	0	0	P_1	$5P_3$	$3P_3$	0	P_4	P_2	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3^-	d_4^-	d_1^+	d_4^+	Solution values b (=X_B)
0	x_2	0	1	1	-1	0	1	0	-1	20
0	x_1	1	0	0	1	0	0	0	0	70
$3P_3$	d_3^-	0	0	-1	1	1	-1	0	1	25
P_4	d_1^+	0	0	0	0	0	1	1	-1	10
z_j-c_j	P_4	0	0	0	0	0	1	0	-1	10
	P_3	0	0	-3	-2	0	-3	0	3	75
	P_2	0	0	0	0	0	0	0	-1	0
	P_1	0	0	-1	0	0	0	0	0	0

In the above table, since all values in P_1 and P_2 row are either zero or negative, the first goal and the second goal are completely achieved. The third goal is not completely attained, because there is a positive value, i.e., 3 in the P_3 row. Since element 3 in P_3 row is above the element –1 in the P_2 row, therefore, the rule is that if there is a positive element at a lower priority level in z_j-c_j, the variable in that column cannot be introduced into the solution, as long as there is a negative element at a higher priority level. Likewise, the positive element 1 in the P_4 row will not be considered.

The optimal solution is:

$x_1 = 70$, $x_2 = 20$, $d_1^+ = 10$, $d_3^- = 25$, $d_1^- = 0$, $d_2^- = 0$

Example 2

Minimize z = $P_1d_1^- + 2P_2d_2^- + P_2d_3^- + P_3d_1^+$

Subject to

$x_1 + x_2 + d_1^- - d_1^+ = 350$
$x_1 + d_2^- = 200$
$x_2 + d_3^- = 300$

$x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0$

Solution:

Substituting $x_1 = 0$, $x_2 = 0$, $d_1^+ = 0$
Therefore, $d_1^- = 350$, $d_2^- = 200$, $d_3^- = 300$

Table 1

	c_j	0	0	P_1	$2P_2$	P_2	P_3	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Solution values b (=X_B)
P_1	d_1^-	1	1	1	0	0	-1	350
$2P_2$	d_2^-	1	0	0	1	0	0	200
P_2	d_3^-	0	1	0	0	1	0	300
z_j-c_j	P_3	0	0	0	0	0	-1	0
	P_2	2	1	0	0	0	0	700
	P_1	1	1	0	0	0	-1	350

Key column = x_1 column
Minimum positive value = Min(350/1, 200/1) = 200
So, d_2^- row is the key row.
Therefore, d_2^- departs & x_1 enters

Table 2

	c_j	0	0	P_1	$2P_2$	P_2	P_3	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Solution values b (=X_B)
P_1	d_1^-	0	1	1	-1	0	-1	150
0	x_1	1	0	0	1	0	0	200
P_2	d_3^-	0	1	0	0	1	0	300
z_j-c_j	P_3	0	0	0	0	0	-1	0
	P_2	0	1	0	-2	0	0	300
	P_1	0	1	0	-1	0	-1	150

Table 3

	c_j	0	0	P_1	$2P_2$	P_2	P_3	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Solution values b ($=X_B$)
0	x_2	0	1	1	-1	0	-1	150
0	x_1	1	0	0	1	0	0	200
P_2	d_3^-	0	0	-1	1	1	1	150
	P_3	0	0	0	0	0	-1	0
z_j-c_j	P_2	0	0	-1	-1	0	1	150
	P_1	0	0	-1	0	0	0	0

Table 4

	c_j	0	0	P_1	$2P_2$	P_2	P_3	
c_B	Basic variables B	x_1	x_2	d_1^-	d_2^-	d_3^-	d_1^+	Solution values b ($=X_B$)
0	x_2	0	1	0	0	1	0	300
0	x_1	1	0	0	1	0	0	200
P_3	d_1^+	0	0	-1	1	1	1	150
	P_3	0	0	-1	1	1	0	150
z_j-c_j	P_2	0	0	0	-2	-1	0	0
	P_1	0	0	-1	0	0	0	0

The optimal solution is:
$x_1 = 200$, $x_2 = 300$, $d_1^- = 0$, $d_2^- = 0$, $d_3^- = 0$, $d_1^+ = 150$.

Self Test Questions

Theory

1. What is goal programming? State clearly its assumptions.

2. Identify the major differences between linear programming and goal programming.

3. State some problem areas in management where goal programming might be applicable.

Practical

1. Company XYZ, produces two products. The maximum sales potential for product 1 and product 2 are 30 units and 40 units respectively. Write the goal constraints for achieving the sales goal by incorporating the deviational variables.

2. An office equipment manufacturer produces two kinds of products, chairs and lamps. Production of either a chair or lamp requires 1 hour of production capacity in the plant. The plant has a maximum production capacity of 10 hours per week. Because of the limited sales capacity, the maximum number of chairs and lamps that can be sold are 6 and 8 per week, respectively. The gross margin from the sale of a chair is Rs. 80 and Rs. 40 for a lamp.

The plant manager has set the following goals arranged in the order of importance:

1. He wants to avoid any underutilization of production capacity.
2. He wants to sell as many chairs and lamps as possible. Since the gross margin from the sale of chair is set a twice the amount of profit from a lamp, he has twice as much desire to achieve the sales goal for chairs as for lamps.
3. He wants to minimize the overtime operation of the plant as much as possible.

Formulate this as a goal programming problem and then solve by both graphical and simplex method.

3. Minimize $z = P_1d_1^- + 2P_2d_2^- + P_2d_3^- + P_3d_1^+$

subject to

$x_1 + x_2 + d_1^- - d_1^+ = 45$
$x_1 + d_2^- = 25$
$x_2 + d_3^- = 35$

$x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0$

4. Minimize $z = P_1d_1^- + 3P_2d_2^- + P_2d_3^- + P_3d_1^+$

subject to

$x_1 + x_2 + d_1^- - d_1^+ = 300$
$x_1 + d_2^- = 140$
$x_2 + d_3^- = 200$

and $x_1, x_2, d_1^-, d_2^-, d_3^-, d_1^+ \geq 0$

Chapter – 9

Game Theory

Introduction

Game theory was developed for the purpose of analyzing competitive situations involving conflicting interests. In other words, game theory is used for decision making under conflicting situations where there are one or more opponents (i.e., players). For example, chess, poker, etc., are the games which have the characteristics of a competition and are played according to definite rules. Game theory provides solutions to such games, assuming that each of the players wants to maximize his profits and minimize his losses.

The game theory models can be classified into several categories. Some important categories are listed below.

- **Two-person & N-person games:** If the number of players is two, it is known as two-person game. On the other hand, if the number of players is N, it is known as N-person game.
- **Zero sum & Non-zero sum game:** In a zero sum game, the sum of the points won equals the sum of the points lost, i.e., one player wins at the expense of the other. To the contrary, if the sum of gains or losses is not equal to zero, it is either positive or negative, then it is known as non-zero sum game. An example of non-zero sum game is the case of two competing firms each with a choice regarding its advertising campaign. In such a situation, both the firms may gain or loose, though their gain or loss may not be equal.
- **Games of Perfect and Imperfect information:** If the strategy of a player can be discovered by his competitor, then it is known as a perfect information game. In case of imperfect information games no player has complete information and tries to guess the real situation.
- **Pure & Mixed strategy games:** If the players select the same strategy each time, then it is referred to as pure strategy games. If a player decides to choose a course of action for each play in accordance with some particularly probability distribution, it is called mixed strategy game.

What are the underlying assumptions of game theory?

- There is finite number of competitors (players).
- The players act reasonably.
- Every player strives to maximize gains and minimize losses.
- Each player has finite number of possible courses of action.

- The choices are assumed to be made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action.
- The pay-off is fixed and predetermined.
- The pay-offs must represent utilities.

Basic Terminology

Player

Each participant (interested party) is called a player.

Strategy

The strategy of a player is the predetermined rule by which a player decides his course of action from the list of courses of action during the game. A strategy may be of two types:

- **Pure strategy.** It is a decision, in advance of all plays, always to choose a particular course of action. In other words, if the best strategy for each player is to play one particular strategy throughout the game, it is called pure strategy.
- **Mixed strategy**. It is a decision, in advance of all plays, to choose a course of action for each play in accordance with some particular probability distribution. In other words, if the optimal plan for each player is to employ different strategies at different times, we call it mixed strategy.

Optimal strategy

The course of action which maximizes the profit of a player or minimizes his loss is called an optimal strategy.

Saddle point

A saddle point is an element of the matrix that is both the smallest element in its row and the largest element in its column. Furthermore, saddle point is also regarded as an equilibrium point in the theory of games.

Pay-off

The outcome of playing the game is called pay-off.

Pay-off Matrix

It is a table showing the outcomes or payoffs of different strategies of the game.

Value of the Game

It refers to the expected outcome per play, when players follow their optimal strategy. It is generally denoted by V.

In the subsequent sections of this chapter, we provide several trivial illustrations. It should be noted that we make no pretense about the realism of these illustrations.

Pure Strategy

The simplest type of game is one where the best strategies for both players are pure strategies. This is the case if and only if, the pay-off matrix contains a saddle point. To illustrate, consider the following pay-off matrix concerning zero sum two person game.

Example

		Player B				
		I	**II**	**III**	**IV**	**V**
Player A	**I**	-2	0	0	5	3
	II	4	2	1	3	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

What is the optimal plan for both the players?

Solution:

We use the maximin (minimax) principle to analyze the game.

		Player B					
		I	**II**	**III**	**IV**	**V**	**Minimum**
Player A	**I**	-2	0	0	5	3	-2
	II	4	2	1	3	2	1
	III	-4	-3	0	-2	6	-4
	IV	5	3	-4	2	-6	-6
Maximum		5	3	1	5	6	

Select minimum from the maximum of columns.
Minimax = 1
Player A will choose II strategy, which yields the maximum payoff of 1.

Select maximum from the minimum of rows.
Maximin = 1
similarly, player B will choose III strategy.

Since the value of maximin coincides with the value of the minimax, therefore, saddle point (equilibrium point) = 1.

The optimal strategies for both players are: Player A must select II strategy and player B must select III strategy. The value of game is 1, which indicates that player A will gain 1 unit and player B will sacrifice 1 unit.

Mixed Strategy

In situations where a saddle point does not exist, the maximin (minimax) principle for solving a game problem breaks down. The concept is illustrated with the help of following example.

Example

Two companies A and B are competing for the same product. Their different strategies are given in the following pay-off matrix:

		Company B		
		I	**II**	**III**
Company A	**I**	-2	14	-2
	II	-5	-6	-4
	III	-6	20	-8

Determine the optimal strategies for both the companies.

Solution:

First, we apply the maximin (minimax) principle to analyze the game.

		Company B			
		I	**II**	**III**	**Minimum**
Company A	**I**	-2	14	-2	-2
	II	-5	-6	-4	-6
	III	-6	20	-8	-8
Maximum		-2	20	-2	

Minimax = -2
Maximin = -2

There are two elements whose value is –2. Hence, the solution to such a game is not unique.

In the above problem, there is no saddle point. In such cases, the maximin and minimax principle of solving a game problem can't be applied. Under this situation, both the companies may resort to what is known as mixed strategy.

A mixed strategy game can be solved by following methods:

- Algebraic Method
- Calculus Method
- Linear Programming Method

Algebraic Method

Consider the zero sum two person game given below:

		Player B	
		I	**II**
Player A	**I**	a	b
	II	c	d

Formulas:

The solution of the game is:

A play's (p, 1 - p)

where:

$$p = \frac{d - c}{(a + d) - (b + c)}$$

B play's (q, 1 - q)

Where:

q = d - b

$$\frac{}{(a + d) - (b + c)}$$

$$\text{Value of the game, } V = \frac{ad - bc}{(a + d) - (b + c)}$$

Example 1

Consider the game of matching coins. Two players, A & B, put down a coin. If coins match (i.e., both are heads or both are tails) A gets rewarded, otherwise B. However, matching on heads gives a double premium. Obtain the best strategies for both players and the value of the game.

	Player B		
Player A		I	II
	I	2	-1
	II	-1	1

Solution:

This game has no **saddle point**.

$$p = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

1 – p = 3/5

$$q = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

1 – q = 3/5

$$V = \frac{2 \text{ X } 1 - (-1) \text{ X } (-1)}{(2 + 1) - (-1 - 1)} = \frac{1}{5}$$

Example 2

Solve the game whose payoff matrix is given below:

	Player B	
Player A	**I**	**II**
I	1	7
II	6	2

Solution:

This game has no **saddle point**.

$$p = \frac{2 - 6}{(1 + 2) - (7 + 6)} = \frac{2}{5}$$

$1 – p = 3/5$

$$q = \frac{2 - 7}{(1 + 2) - (7 + 6)} = \frac{1}{2}$$

$1 – q = 1/2$

$$V = \frac{1 \text{ X } 2 - (7 \text{ X } 6)}{(1 + 2) - (7 + 6)} = 4$$

Calculus Method

This method is almost similar to the previous method except that instead of equating the two expected values, the expected value for a given player is maximized.

Consider the zero sum two person game given below:

	Player B	
Player A	**I**	**II**
I	a	b
II	c	d

Formulas:

The solution of the game is:

A play's (p, 1 - p)

Where:

$$p = \frac{d - c}{(a + d) - (b + c)}$$

B play's (q, 1 - q)

Where:

$$q = \frac{d - b}{(a + d) - (b + c)}$$

Value of the game, V = apq + c(1 – p)q + bp(1 – q) + d(1 – p)(1 – q)

To illustrate this method, consider the same example discussed in the previous section.

Example 1

Consider the following game:

		Player B	
		I	**II**
Player A	**I**	2	-1
	II	-1	1

Solution:

This game has no **saddle point**.

$$p = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

1 – p = 3/5

$$q = \frac{1 - (-1)}{(2 + 1) - (-1 - 1)} = \frac{2}{5}$$

1 – q = 3/5

V = 2 X 2/5 X 2/5 + (-1) X 3/5 X 2/5 + (-1) X 2/5 X 3/5 + 1 X 3/5 X 3/5 = 1/5

Example 2

Solve the game whose pay-off matrix is given below:

		Player B	
		I	**II**
Player A	**I**	1	3
	II	5	2

Solution:

This game has no **saddle point**.

$$p = \frac{2 - 5}{(1 + 2) - (3 + 5)} = \frac{3}{5}$$

1 – p = 2/5

$$q = \frac{2 - 3}{(1 + 2) - (3 + 5)} = \frac{1}{5}$$

1 – q = 4/5

V = 1 X 3/5 X 1/5 + 5 X 2/5 X 1/5 + 3 X 3/5 X 4/5 + 2 X 2/5 X 4/5 = 13/5

Linear Programming Method

The linear programming technique is used for solving mixed strategy games of dimensions greater than (2 X 2) size. The following simple example is used to explain the procedure.

Example

Two companies are competing for the same product. To improve its market share, company A decides to launch the following strategies.

- A1 = give discount coupons
- A2 = home delivery services
- A3 = free gifts

The company B decides to use media advertising to promote its product.

- B1 = internet
- B2 = newspaper
- B3 = magazine

		Company B		
		B1	**B2**	**B3**
Company A	**A1**	3	-4	2
	A2	1	-7	-3
	A3	-2	4	7

Use linear programming to determine the best strategies for both the companies.

Solution:

		Company B			Minimum
		B1	**B2**	**B3**	
Company A	**A1**	3	-4	2	-4
	A2	1	-7	-3	-7
	A3	-2	4	7	-2
Maximum		3	4	7	

Minimax = -2
Maximin = 3

This game has no **saddle point**. So the value of the game lies between –2 and +3. It is possible that the value of game may be negative or zero. Thus, a constant k is added to all the elements of pay-off matrix. Let k = 3, then the given pay-off matrix becomes:

	Company B			
		B1	B2	B3
Company A	A1	6	-1	5
	A2	4	-4	0
	A3	1	7	10

Let
V = value of the game
p1, p2 & p3 = probabilities of selecting strategies A1, A2 & A3 respectively.
q1, q2 & q3 = probabilities of selecting strategies B1, B2 & B3 respectively.

		Company B			Probability
		B1	B2	B3	
Company A	A1	6	-1	5	p1
	A2	4	-4	0	p2
	A3	1	7	10	p3
Probability		q1	q2	q3	

Company A's objective is to maximize the expected gains, which can be achieved by maximizing V, i.e., it might gain more than V if company B adopts a poor strategy. Hence, the expected gain for company A will be as follows:

$6p1 + 4p2 + p3 \geq V$
$-p1 - 4p2 + 7p3 \geq V$
$5p1 + 0p2 + 10p3 \geq V$
$p1 + p2 + p3 = 1$
and $p1, p2, p3 \geq 0$

Dividing the above constraints by V, we get
$6p1/V + 4p2/V + p3/V \geq 1$
$-p1/V - 4p2/V + 7p3/V \geq 1$
$5p1/V + 0p2/V + 10p3/V \geq 1$
$p1/V + p2/V + p3/V = 1/V$

To simplify the problem, we put
$p1/V = x_1$, $p2/V = x_2$, $p3/V = x_3$

In order to maximize V, company A can
Minimize $1/V = x_1 + x_2 + x_3$
subject to

$6x_1 + 4x_2 + x_3 \geq 1$
$-x_1 - 4x_2 + 7x_3 \geq 1$
$5x_1 + 0x_2 + 10x_3 \geq 1$

and $x_1, x_2, x_3 \geq 0$

Company B's objective is to minimize its expected losses, which can be reduced by minimizing V, i.e., company A adopts a poor strategy. Hence, the expected loss for company B will be as follows:

$6q1 - q2 + 5q3 \leq V$
$4q1 - 4q2 + 0q3 \leq V$
$q1 + 7q2 + 10q3 \leq V$
$q1 + q2 + q3 = 1$

and $q1, q2, q3 \geq 0$

Dividing the above constraints by V, we get
$6q1/V - q2/V + 5q3/V \leq 1$
$4q1/V - 4q2/V + 0q3/V \leq 1$
$q1/V + 7q2/V + 10q3/V \leq 1$
$q1/V + q2/V + q3/V = 1/V$

To simplify the problem, we put
$q1/V = y_1$, $q2/V = y_2$, $q3/V = y_3$

In order to minimize V, company B can
Maximize $1/V = y_1 + y_2 + y_3$
subject to

$6y_1 - y_2 + 5y_3 \leq 1$
$4y_1 - 4y_2 + 0y_3 \leq 1$
$y_1 + 7y_2 + 10y_3 \leq 1$

and $y_1, y_2, y_3 \geq 0$

To solve this problem, we introduce slack variables to convert inequalities to equalities. The problem becomes

Maximize $y_1 + y_2 + y_3 + 0y_4 + 0y_5 + 0y_6$

$6y_1 - y_2 + 5y_3 + y_4 = 1$
$4y_1 - 4y_2 + y_5 = 1$
$y_1 + 7y_2 + 10y_3 + y_6 = 1$

Initial Basic Feasible Solution

$y_1 = 0, y_2 = 0, y_3 = 0, z = 0$
$y_4 = 1, y_5 = 1, y_6 = 1$

Table 1

	c_j	1	1	1	0	0	0	
c_B	Basic Variables B	y_1	y_2	y_3	y_4	y_5	y_6	Solution values b (=X_B)
0	y_4	6	-1	5	1	0	0	1
0	y_5	4	-4	0	0	1	0	1
0	y_6	1	7	10	0	0	1	1
z_j-c_j		-1	-1	-1	0	0	0	

Key column = y_1 column
Minimum positive value = 1/6
Key row = y_4 row
Pivot element = 6
y_4 departs and y_1 enters.

Table 2

	c_j	1	1	1	0	0	0	
c_B	Basic Variable B	y_1	y_2	y_3	y_4	y_5	y_6	Solution values b (=X_B)
1	y_1	1	-1/6	5/6	1/6	0	0	1/6
0	y_5	0	-10/3	-10/3	-2/3	1	0	1/3
0	y_6	0	43/6	55/6	-1/6	0	1	5/6
z_j-c_j		0	-7/6	-1/6	1/6	0	0	

Final Table

	c_j	1	1	1	0	0	0	
c_B	**Basic Variable B**	y_1	y_2	y_3	y_4	y_5	y_6	**Solution values b (=X_B)**
1	y_1	1	0	45/43	7/43	0	1/43	8/43
0	y_5	0	0	40/43	-32/43	1	20/43	31/43
1	y_2	0	1	55/43	-1/43	0	6/43	5/43
z_j-c_j		0	0	57/43	6/43	0	7/43	

The values for y_1, y_2 & y_3 are 8/43, 5/43 & 0 respectively.

I/V = $y_1 + y_2 + y_3$ = 8/43 + 5/43 + 0 = 13/43
or V = 43/13

Company B's optimal strategy
q1 = V X y_1 = 43/13 X 8/43 = 8/13
q2 = V X y_2 = 43/13 X 5/43 = 5/13
q3 = V X y_3 = 43/13 X 0 = 0

Hence, company B's optimal strategy is (8/13, 5/13, 0).

Company A's optimal strategy

The values for x_1, x_2 & x_3 can be obtained from the final simplex table.

x_1 = 6/43, x_2 = 0 & x_3 = 7/43
p1 = V X x_1 = 43/13 X 6/43 = 6/13
p2 = V X x_2 = 43/13 X 0 = 0
p3 = V X x_3 = 43/13 X 7/43 = 7/13

Hence, company A's optimal strategy is (6/13, 0, 7/13).

Dominance

The principle of dominance states that if one strategy of a player dominates over the other strategy in all conditions then the later strategy can be ignored. A strategy dominates over the other only if it is preferable over other in all conditions. The concept of dominance is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

Rules

- If all the elements of a column (say i_{th} column) are greater than or equal to the corresponding elements of any other column (say j_{th} column), then the i_{th} column is dominated by the j_{th} column and can be deleted from the matrix.
- If all the elements of a row (say i_{th} row) are less than or equal to the corresponding elements of any other row (say j_{th} row), then the i_{th} row is dominated by the j_{th} row and can be deleted from the matrix.

Example

		Player B			
		I	**II**	**III**	**IV**
Player A	**I**	3	5	4	2
	II	5	6	2	4
	III	2	1	4	0
	IV	3	3	5	2

Use the concept of dominance to solve this problem.

Solution:

		Player B				
		I	**II**	**III**	**IV**	**Minimum**
Player A	**I**	3	5	4	2	2
	II	5	6	2	4	2
	III	2	1	4	0	0
	IV	3	3	5	2	2
Maximum		5	6	5	4	

There is no **saddle point** in this game.

Using dominance property

If a column is greater than another column (compare corresponding elements), then delete that column.
Here, I and II column are greater than the IV column. So, player B has no incentive in using his I and II course of action.

	Player B		
Player A		III	IV
	I	4	2
	II	2	4
	III	4	0
	IV	5	2

If a row is smaller than another row (compare corresponding elements), then delete that row.
Here, I and III row are smaller than IV row. So, player A has no incentive in using his I and III course of action.

	Player B		
Player A		III	IV
	II	2	4
	IV	5	2

Now you can use any one of the following to determine the value of game

- Algebraic Method
- Calculus Method
- Linear Programming Method

2 x n Games

Games where one player has only two courses of action while the other has more than two, are called 2 X n or n X 2 games. If these games do not have a saddle point or are reducible by the dominance method, then before solving these games we write all 2 X 2 sub-games and determine the value of each 2 X 2 sub-game. This method is illustrated by the following example.

Example

Determine the solution of game for the pay-off matrix given below:

	Player B			
Player A		I	II	III
	I	-3	-1	7
	II	4	1	-2

Solution:

Obviously, there is no saddle point and also no course of action dominates the other. Therefore, we consider each 2 X 2 sub-game and obtain their values.

(a)

	Player B		
Player A		I	II
	I	-3	-1
	II	4	1

The saddle point is 1. So the value of game, V1 is 1.

(b)

	Player B		
Player A		I	II
	I	-3	7
	II	4	-2

This game has no saddle point, so we use the algebraic method.

$$\text{Value of game, V2} = \frac{(-3) \text{ X } (-2) - (7 \text{ X } 4)}{(-3 - 2) - (7 + 4)} = \frac{11}{8}$$

(c)

	Player B		
Player A		II	III
	I	-1	7
	II	1	-2

This game has no saddle point, so we use the algebraic method.

$$\text{Value of game, V3} = \frac{(-1) \text{ X } (-2) - (7 \text{ X } 1)}{(-1 - 2) - (7 + 1)} = \frac{5}{11}$$

The 2 X 2 sub-game with the lowest value is (c) and hence the solution to this game provides the solution to the larger game.

Using algebraic method:

A plays (3/11, 8/11)
B plays (0, 9/11, 2/11)
Value of game is 5/11.

Graphical Method

The method discussed in the previous section is feasible when the value of *n* is small, because the larger value of *n* will yield a larger number of 2 X 2 sub-games. In this section, we discuss another method for solving 2 X n games. This method can only be used in games with no saddle point, and having a pay-off matrix of type n X 2 or 2 X n.

Example

Consider the following pay-off matrix

		Player B	
		B_1	**B_2**
Player A	**A_1**	-2	4
	A_1	8	3
	A_1	9	0

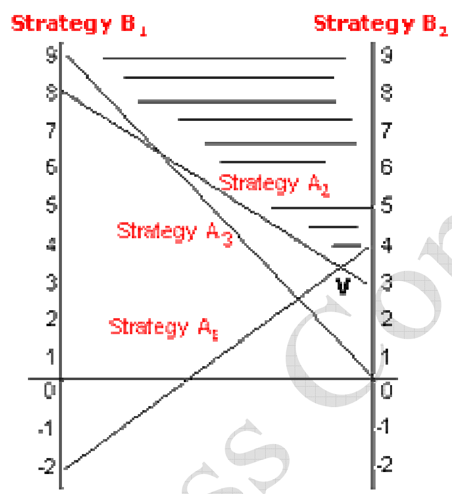
Solution:

The game does not have a saddle point as shown in the following table.

		Player B		Minimum	Probability
		B_1	**B_2**		
Player A	**A_1**	-2	4	-2	q_1
	A_2	8	3	3	q_2
	A_3	9	0	0	q_3
Maximum		9	4		
Probability		p_1	p_1		

Maximin = 4, Minimax = 3

First, we draw two parallel lines 1 unit distance apart and mark a scale on each. The two parallel lines represent strategies of player B.
If player A selects strategy A1, player B can win –2 (i.e., loose 2 units) or 4 units depending on B's selection of strategies. The value -2 is plotted along the vertical axis under strategy B_1 and the value 4 is plotted along the vertical axis under strategy B_2. A straight line joining the two points is then drawn.
Similarly, we can plot strategies A_2 and A_3 also. The problem is graphed in the following figure.



The lowest point V in the shaded region indicates the value of game. From the above figure, the value of the game is 3.4 units. Likewise, we can draw a graph for player B.

The point of optimal solution (i.e., maximin point) occurs at the intersection of two lines:

$E1 = -2p_1 + 4p_2$ and
$E2 = 8p_1 + 3p_2$

Comparing the above two equations, we have

$-2p_1 + 4p_2 = 8p_1 + 3p_2$

Substituting $p_2 = 1 - p_1$
$-2p_1 + 4(1 - p_1) = 8p_1 + 3(1 - p_1)$
$p_1 = 1/11$
$p_2 = 10/11$

Substituting the values of p_1 and p_2 in equation E1

V = -2 (1/11) + 4 (10/11) = 3.4 units

Bidding Problems

Several competitive situations involve bidding for contracts, tenders, licenses, etc. All the bidding problems may be classified into two groups:

- Auction Bids - In case of auction bids, the bids are open.
- Closed Bids - In case of closed bids, each bidder submits his bid in a closed envelope.

1. Auction Bids or Open Bids

Example

A chair and a table worth Rs. 80 and Rs.120 are to be auctioned at a public sale. There are only two bidders - Vinay and Manish. Vinay and Manish have Rs. 100 and Rs. 130 respectively. What should be their strategies, if each bidder is interested in maximizing his return? Assume that both the bidders have complete information about each other's money position.

Solution:

Suppose the bid increases successively by the amount Rs. Δ. It should be noted that at any bid, each player has a option to increase the bid or to leave the opponent's bid stand.

Suppose Manish has bid of Rs. x on the chair.

Case I

If Vinay allows Manish to win the chair for Rs. x then Manish will have only (130 - x) rupees for bidding on the table. Thus, he can not make his bid for the table more than Rs. 130 - x. Hence, Vinay will definitely win the table in Rs. $(130 - x + \Delta)$. Thus, Vinay's gain when he allows Manish to win the chair for Rs. x is

Rs. $[120 - (130 - x + \Delta)]$
$= $ Rs. $(x - \Delta - 10)$

Case II

To the contrary, if Vinay bids Rs. $x + \Delta$ for the chair and Manish allows him to win at this bid, then Vinay's gain is
Rs. $[80 - (x + \Delta)]$
$= 80 - x - \Delta$

Now since Vinay wants to maximize his return, he should bid x + Δ for the chair provided
$80 - x - \Delta \geq x - \Delta - 10$
$= 2x \leq 90$
$= x \leq 45$

Thus, till $x \leq 45$, Vinay should bid for chair. When $x > 45$, he should allow Manish to win the chair for that bid.

Likewise, in the two cases, Manish's gains are
$[120 - (100 - y) - \Delta]$ and $[80 - (y + \Delta)]$
Where y is the Vinay's bid for the chair

Thus, Manish should bid y + Δ for the chair provided

$80 - (y + \Delta) \geq 120 - (100 - y) - \Delta$
$= y \leq 30$

Obviously, Vinay will take the chair in Rs. 30 because he can increase his bid without any loss upto Rs. 45, Manish will take the table in Rs. (100 - 30) = 70 because Vinay, after winning the chair for Rs. 30 cannot increase his bid for the table more than Rs. 70. Thus, Manish will get the table for Rs. 70. The gain of Vinay is Rs. (80 - 30) = Rs. 50, and of Manish is Rs. (120 - 70) = Rs. 50.

2. Closed Bids

Example

A joystick and a keyboard worth Rs 80. and Rs. 100 are to be bid simultaneously by two bidders A and B. Both have intention of devoting a total sum of Rs. 110 for the items. If each uses a minimax criterion, find the resulting bids.

Solution.

Since the bids are to be made simultaneously, they are closed bids.

Suppose P and Q are A's best bids for the joystick and the keyboard respectively. A's best bids are those which give the same profit to A on both the items. If t is the total profit associated with a successful bid, then

$2t = (80 - P) + (100 - Q)$
$2t = 180 - (P + Q)$

$2t = 180 - 110$
$2t = 70$
$t = 35$

$P = 80 - t$
$= 80 - 35$
$= 45$
and

$Q = 100 - t$
$= 100 - 35$
$= 65.$

Thus, optimal bids for A are Rs. 45 for joystick and Rs. 65 for keyboard.
The optimal bids for B will be same as A's optimal bids.

Advantages & Limitations of Game Theory

Advantages

- Game theory gives insight into several less-known aspects, which arise in situations of conflicting interests. For example, it describes and explains the phenomena of bargaining and coalition-formation.
- Game theory develops a framework for analyzing decision making in such situations where interdependence of firms is considered.
- At least in two-person zero-sum games, game theory outlines a scientific quantitative technique that can be used by players to arrive at an optimal strategy.

Limitations

- The assumption that players have the knowledge about their own pay-offs and pay-offs of others is not practical.
- The techniques of solving games involving mixed strategies particularly in case of large pay-off matrix is very complicated.
- All the competitive problems cannot be analyzed with the help of game theory.

Self Test Questions

Theory

1. What is 'two-person zero-sum game' ?

2. Define the following:

- Saddle point
- Pure strategy.
- Pay-off matrix.
- Optimal strategies.

3. Explain briefly the importance of the principle of dominance.

4. What are the advantages & limitations of game theory?

Practical

1. Consider the game whose pay-off matrix is given below. Find its solution.

		Player B		
		I	II	III
Player A	I	-4	-6	3
	II	-3	-3	6
	III	2	-3	4

2. Two companies A and B are competing for the same product. Their different strategies are given in the following pay-off matrix:

		Company B		
		I	II	III
Company A	I	2	-2	3
	II	-3	5	-1

Determine the best strategies and find the value of the game.

3. Solve the following games:

(a)

		Player B				
		I	II	III	IV	V
Player A	I	4	0	1	7	-1
	II	0	-3	-5	-7	5
	III	3	2	3	4	3
	IV	-6	1	-1	0	5
	V	0	0	6	0	0

(b

		Player B		
		I	II	III
Player A	I	-2	15	-2
	II	-5	-6	-4
	III	-5	20	-8

(c)

		Player B		
		I	II	III
Player A	I	2	-1	3
	II	2	-1	2
	III	-1	0	0
	IV	2	0	4

(d)

		Player B			
		I	II	III	IV
Player A	I	-5	3	1	20
	II	5	5	4	6
	III	-4	-2	0	-5

(e)

		Player B			
		I	II	III	IV
Player A	I	3	-5	0	6
	II	-4	-2	1	2
	III	5	4	2	3

4. Use the dominance principle to solve the following game:

0	0	0	0	0	0
4	2	0	2	1	1
4	3	1	3	2	2
4	3	7	-5	1	2
4	3	4	-1	2	2
4	3	3	-2	2	2

5. Solve the following games:

(a)

		Player B		
		I	**II**	**III**
Player A	**I**	-5	-1	-1
	II	4	0	2
	III	-5	2	0

(b)

	Player B			
Player A	6	8	3	13
	4	1	5	3
	8	10	4	12
	3	6	7	12

6. Solve the following game algebraically

		Player B		
		I	**II**	**III**
Player A	**I**	4	2	4
	II	2	4	0
	III	4	0	8

7. Reduce each of the following games by using the rule of dominance and then solve the reduced game by any of the method you have studied:

(a)

	B1	**B2**	**B3**

A1	3	8	5
A2	6	2	7
A3	4	5	6

(b)

	B1	**B2**	**B3**	**B4**	**B5**
A1	8	7	6	-1	2
A2	12	10	12	0	4
A3	14	6	8	14	16

8. Two players A & B, without showing each other put a coin on a table with head or tail up. If the coins show the same side (both head or tail), the player A takes both the coins, otherwise B gets them. Construct the matrix of the game and solve it.

9. In a game of matching coins with two players, suppose A wins one unit of the value when there are two heads; wins nothing when there are two tails and loses 1/2 units of value when there is one head and one tail. Determine the pay off matrix, the optimal strategies for both the players.

Chapter – 10

Waiting Line Models

Introduction

Waiting lines are the most frequently encountered problems in everyday life. For example, queue at a cafeteria, library, bank, etc. Common to all of these cases are the arrivals of objects requiring service and the attendant delays when the service mechanism is busy. Waiting lines cannot be eliminated completely, but suitable techniques can be used to reduce the waiting time of an object in the system. A long waiting line may result in loss of customers to an organization. Waiting time can be reduced by providing additional service facilities, but it may result in an increase in the idle time of the service mechanism.

Definition

Queuing theory is based on mathematical theories and deals with the problems arising due to flow of customers towards the service facility.

The waiting line models help the management in balancing between the cost associated with waiting and the cost of providing service. Thus, queuing or waiting line models can be applied in such situations where decisions have to be taken to minimize the waiting time with minimum investment cost.

Basic Terminology

The present section focuses on the standard vocabulary of Waiting Line Models.

Queuing Model

It is a suitable model used to represent a service oriented problem, where customers arrive randomly to receive some service, the service time being also a random variable.

Arrival

The statistical pattern of the arrival can be indicated through the probability distribution of the number of the arrivals in an interval.

Service Time

The time taken by a server to complete service is known as service time.

Server

It is a mechanism through which service is offered.

Queue Discipline

It is the order in which the members of the queue are offered service.

Poisson Process

It is a probabilistic phenomenon where the number of arrivals in an interval of length t follows a Poisson distribution with parameter λt, where λ is the rate of arrival.

Queue

A group of items waiting to receive service, including those receiving the service, is known as queue.

Waiting time in queue

Time spent by a customer in the queue before being served.

Waiting time in the system

It is the total time spent by a customer in the system. It can be calculated as follows:

Waiting time in the system = Waiting time in queue + Service time

Queue length

Number of persons in the system at any time.

Average length of line

The number of customers in the queue per unit of time.

Average idle time

The average time for which the system remains idle.

FIFO

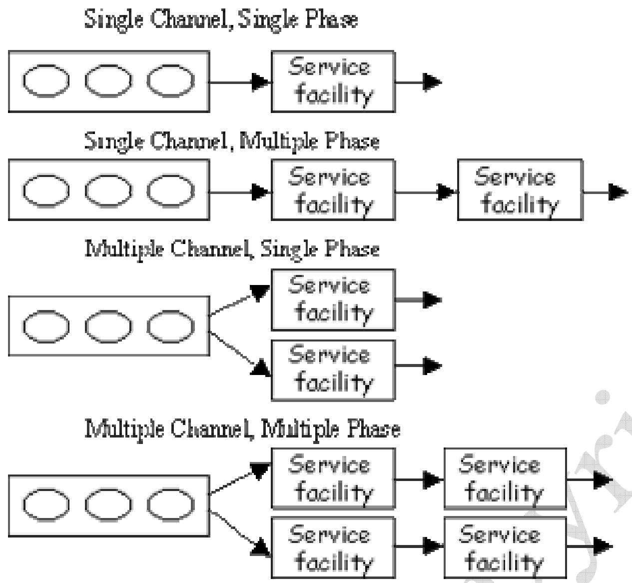
It is the first in first out queue discipline.

Bulk Arrivals

If more than one customer enter the system at an arrival event, it is known as bulk arrivals.
Please note that bulk arrivals are not embodied in the models of the subsequent sections.

Components of Queuing System

1. **Input Source:** The input source generates customers for the service mechanism. The most important characteristic of the input source is its size. It may be either finite or infinite. Please note that the calculations are far easier for the infinite case, therefore, this assumption is often made even when the actual size is relatively large. If the population size is finite, then the analysis of queuing model becomes more involved.
   The statistical pattern by which calling units are generated over time must also be specified. It may be Poisson or Exponential probability distribution.
2. **Queue:** It is characterized by the maximum permissible number of units that it can contain. Queues may be infinite or finite.
3. **Service Discipline:** It refers to the order in which members of the queue are selected for service. Frequently, the discipline is first come, first served.
   Following are some other disciplines:
   - LIFO (Last In First Out)
   - SIRO (Service In Random Order)
   - Priority System
4. **Service Mechanism:** A specification of the service mechanism includes a description of time to complete a service and the number of customers who are satisfied at each service event. The service mechanism also prescribes the number and configuration of servers. If there is more than one service facility, the calling unit may receive service from a sequence of these. At a given facility, the unit enters one of the parallel service channels and is completely serviced by that server. Most elementary models assume one service facility with either one or a finite number of servers.The following figure shows the physical layout of service facilities.



Unusual Customer/Server Behavior

Customer's Behavior

- **Balking.** A customer may not like to join the queue due to long waiting line.
- **Reneging.** A customer may leave the queue after waiting for sometime due to **impatience**.
- **Collusion.** Several customers may cooperate and only one of them may stand in the queue.
- **Jockeying.** When there are a number of queues, a customer may move from one queue to another in hope of receiving the service quickly.

Server's Behavior

- **Failure.** The service may be interrupted due to failure of a server (machinery).
- **Changing service rates.** A server may speed up or slow down, depending on the number of customers in the queue. For example, when the queue is long, a server may speed up in response to the pressure. On the contrary, it may slow down if the queue is very small.
- **Batch processing.** A server may service several customers simultaneously, a phenomenon known as batch processing.

What are the underlying assumptions?

- The source population has infinite size.
- The inter-arrival time has an exponential probability distribution with a mean arrival rate of l customer arrivals per unit time.
- There is no unusual customer behaviour.
- The service discipline is FIFO.
- The service time has an exponential probability distribution with a mean service rate of m service completions per unit time.
- The mean arrival rate is less than the mean service rate, i.e., l < m.
- There is no unusual server behaviour.

The M/M/1 (∝/FIFO) system

It is a queuing model where the arrivals follow a Poisson process, service times are exponentially distributed and there is only one server. In other words, it is a system with Poisson input, exponential waiting time and Poisson output with single channel.

Queue capacity of the system is infinite with first in first out mode. The first M in the notation stands for Poisson input, second M for Poisson output, 1 for the number of servers and ∝ for infinite capacity of the system.

Formulas

Probability of zero unit in the queue (P_o) = $1 - \dfrac{\lambda}{\mu}$

Average queue length (L_q) = $\dfrac{\lambda^2}{\mu(\mu - \lambda)}$

Average number of units in the system (L_s) = $\dfrac{\lambda}{\mu - \lambda}$

Average waiting time of an arrival (W_q) = $\dfrac{\lambda}{\mu(\mu - \lambda)}$

Average waiting time of an arrival in the system (W_s) = $\dfrac{1}{\mu - \lambda}$

Example 1

Students arrive at the head office of www.universalteacher.com according to a Poisson input process with a mean rate of 40 per hour. The time required to serve a student has an exponential distribution with a mean of 50 per hour. Assume that the students are served by a single individual; find the average waiting time of a student.

Solution:

Given
λ = 40/hour, μ = 50/hour

Average waiting time of a student before receiving service (W_q) = $\dfrac{40}{50(50 - 40)}$ = 4.8 minutes

Example 2

New Delhi Railway Station has a single ticket counter. During the rush hours, customers arrive at the rate of 10 per hour. The average number of customers that can be served is 12 per hour. Find out the following:

- Probability that the ticket counter is free.
- Average number of customers in the queue.

Solution:

Given
λ = 10/hour, μ = 12/hour

Probability that the counter is free = $1 - \dfrac{10}{12}$ = 1/6

Average number of customers in the queue (L_q) = $\dfrac{(10)^2}{12\,(12 - 10)}$ = 25/6

Example 3

At Bharat petrol pump, customers arrive according to a Poisson process with an average time of 5 minutes between arrivals. The service time is exponentially distributed with mean time = 2 minutes. On the basis of this information, find out

1. What would be the average queue length?
2. What would be the average number of customers in the queuing system?
3. What is the average time spent by a car in the petrol pump?
4. What is the average waiting time of a car before receiving petrol?

Solution:

Average inter arrival time = $\frac{1}{\lambda}$ = 5minutes = $\frac{1}{12}$ hour

λ = 12/hour

Average service time = $\frac{1}{\mu}$ = 2 minutes = $\frac{1}{30}$ hour

μ = 30/hour

Average queue length, L_q = $\frac{(12)^2}{30(30 - 12)}$ = $\frac{4}{15}$

Average number of customers, L_s = $\frac{12}{30 - 12}$ = $\frac{2}{3}$

Average time spent at the petrol pump = $\frac{1}{30 - 12}$ = 3.33 minutes

Average waiting time of a car before receiving petrol = $\frac{12}{30(30 - 12)}$ = 1.33 minutes

Example 4

Universal Bank is considering opening a drive in window for customer service. Management estimates that customers will arrive at the rate of 15 per hour. The teller whom it is considering to staff the window can service customers at the rate of one every three minutes.

Assuming Poisson arrivals and exponential service find

1. Average number in the waiting line.
2. Average number in the system.
3. Average waiting time in line.
4. Average waiting time in the system.

Solution:

Given
λ = 15/hour,
μ = 3/60 hour
or 20/hour

Average number in the waiting line = $\dfrac{(15)^2}{20(20 - 15)}$ = 2.25 customers

Average number in the system = $\dfrac{15}{20 - 15}$ = 3 customers

Average waiting time in line = $\dfrac{15}{20(20 - 15)}$ = 0.15 hours

Average waiting time in the system = $\dfrac{1}{20 - 15}$ = 0.20 hours

Example 5

Chhabra Saree Emporium has a single cashier. During the rush hours, customers arrive at the rate of 10 per hour. The average number of customers that can be processed by the cashier is 12 per hour. On the basis of this information, find the following:

- Probability that the cashier is idle
- Average number of customers in the queuing system
- Average time a customer spends in the system
- Average number of customers in the queue
- Average time a customer spends in the queue

Solution:

Given

$\lambda = 10/\text{hour}$, $\mu = 12/\text{hour}$

$$P_o = 1 - \frac{10}{12} = 1/6$$

$$L_s = \frac{10}{12 - 10} = 5 \text{ customers}$$

$$W_s = \frac{1}{12 - 10} = 30 \text{ minutes}$$

$$L_q = \frac{(10)^2}{12(12 - 10)} = 25/6 \text{ customers}$$

$$W_q = \frac{10}{12(12 - 10)} = 25 \text{ minutes}$$

The M/M/1 (N/FIFO) system

It is a queuing model where the arrivals follow a Poisson process, service times are exponentially distributed and there is only one server. Capacity of the system is limited to N with first in first out mode.

The first M in the notation stands for Poisson input, second M for Poisson output, 1 for the number of servers and N for capacity of the system.

$\rho = \lambda/\mu$

$$P_o = \frac{1-\rho}{1-\rho^{N+1}}$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$L_q = L_s - \lambda/\mu$$

$$W_q = \frac{L_q}{\lambda}$$

$$W_s = \frac{L_s}{\lambda}$$

Example

Students arrive at the head office of www.universalteacher.com according to a Poisson input process with a mean rate of 30 per day. The time required to serve a student has an exponential distribution with a mean of 36 minutes. Assume that the students are served by a single individual, and queue capacity is 9. On the basis of this information, find the following:

- The probability of zero unit in the queue.
- The average line length.

Solution:

$$\lambda = \frac{30}{60 \text{ X } 24}$$

= 1/48 students per minute

μ = 1/36 students per minute

ρ = 36/48 = 0.75

N = 9

$$P_o = \frac{1- 0.75}{1- (0.75)^{9+1}}$$

= 0.26

$$L_s = \frac{0.75}{1 - 0.75} - \frac{(9 + 1)(0.75)^{9+1}}{1- (0.75)^{9+1}}$$

= 2.40 or 2 students.

The M/M/C ($\propto$/FIFO) system

It is a queuing model where the arrivals follow a Poisson process, service times are exponentially distributed and there are C servers.

$$\frac{1}{P_0} = \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \times \frac{1}{1-\rho}$$

Where $\rho = \dfrac{\lambda}{c\mu}$

$$L_q = P_0 \times \frac{(\lambda/\mu)^c}{c!} \times \frac{\rho}{(1-\rho)^2}$$

$$W_q = \frac{1}{\lambda} \times L_q$$

$$W_s = W_q + \frac{1}{\mu}$$

$$L_s = L_q + \frac{\lambda}{\mu}$$

Example 1

The Silver Spoon Restaurant has only two waiters. Customers arrive according to a Poisson process with a mean rate of 10 per hour. The service for each customer is exponential with mean of 4 minutes. On the basis of this information, find the following:

- The probability of having to wait for service.
- The expected percentage of idle time for each waiter.

Solution.

This is an example of M/M/C, where c = 2

λ = 10 per hour or 1/6 per minute.
μ = 1/4 per minute
ρ = 1/3

$$\frac{1}{P_0} = \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^c}{c!} \times \frac{1}{1-\rho}$$

$$\frac{1}{P_0} = \sum_{n=0}^{1} \frac{(2/3)^n}{n!} + \frac{(2/3)^2}{2!} \times \frac{1}{1-1/3}$$

$$\frac{1}{P_0} = 1 + \frac{2}{3} + \frac{1}{3}$$

$$P_0 = \frac{1}{2}$$

The expected percentage of idle time for each waiter.

$1 - \rho = \quad 1 - 1/3 = 2/3 = 67\%$

Example 2

Universal Bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service times distributions for both deposits and withdrawals are exponential with mean service time 2 minutes per customer. Deposits & withdrawals are found to arrive in a Poisson fashion with mean arrival rate 20 per hour. What would be the effect on the average waiting time for depositors and withdrawers, if each teller could handle both withdrawers & depositors?

Solution:

Given

λ = 20 per hour or 1/3 per minute, μ = 1/2 per minute, c = 2

Case I - Treating depositors and withdrawers as unit of M/M/1 system.

Average waiting time of an arrival (W_q) =

$$\frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_q = \frac{1/3}{1/2\,(1/2 - 1/3)} = 4 \text{ minutes}$$

Case II - If each teller handles both depositors and withdrawers.

$P_0 = 1/2$

$L_q = 1/12$

$$W_q = \frac{1}{\lambda} \times L_q$$

$W_q =$ 1/4 minutes

Hence, when both tellers handle both withdrawals & deposits, then expected waiting time is reduced.

The M/E_k/1 (∝/FIFO) system

It is a queuing model where the arrivals follow a Poisson process, service time follows an Erlang (k) probability distribution and the number of server is one.

Queue capacity of the system is infinite with first in first out mode. The first M in the notation stands for Poisson input, k for number of phases, 1 for the number of servers and ∝ for infinite capacity of the system.

An Erlang Family (E_k)

E_k of a probability distribution is the probability distribution of a random variable, which can be expressed as the sum 'k' independently, identically distributed exponential variables.

The expected numbers of customers in the queue, $L_q = \dfrac{(1+k)}{2k} \times \dfrac{\lambda^2}{\mu(\mu-\lambda)}$

The expected waiting time before being served, $W_q = \dfrac{(1+k)}{2k} \times \dfrac{\lambda}{\mu(\mu-\lambda)}$

The expected time spent in the system, $W_s = W_q + \dfrac{1}{\mu}$

The expected numbers of customers in the system, L_s = λW_s

Example 1

The registration of a student at www.universalteacher.com requires three steps to be completed sequentially. The time taken to perform each step follows an exponential distribution with mean 30/3 minutes and is independent of each other. Students arrive at the head office according to a Poisson input process with a mean rate of 25 per hour. Assuming that there is only one person for registration. On the basis of this information, find the following:

a. expected waiting time
b. Expected numbers of students in the queue.

Solution:

This is an $M/E_k/1$ system.

Here k = 3, λ = 25 per hour.

Service time per phase = $\frac{1}{3\mu} = \frac{30}{3}$

Therefore, μ = 30 per hour.

The expected numbers of students in the queue, L_q = $\frac{1+3}{2 \times 3} \times \frac{(25)^2}{30(30-25)}$ = 2.78 students or 3 students

The expected waiting time before being served, W_q = $\frac{1+3}{2 \times 3} \times \frac{25}{30(30-25)}$ = 1/9 hour or 6.67 minutes

Example 2

Repair of a certain type of machine requires three steps to be completed sequentially. The time taken to perform each step follows an exponential distribution with mean 20/3 minutes and is independent of each other. The machine breakdown follows a Poisson process with rate of 1per 2 hours. Assuming that there is only one repairman, find out

a. The expected idle time of a machine.
b. The average waiting time of a broken down machine in a queue.

c. The expected number of broken down machines in the queue.
d. The average number of machines which are not in operation

Solution:

This is an $M/E_k/1$ system.

Here k =3, λ =1/2 per hour.

Service time per phase = $\dfrac{1}{3\mu} = \dfrac{20}{3}$

Therefore, μ = 3 per hour.

The expected numbers of customers in the queue, L_q = $\dfrac{1+3}{2 \times 3} \times \dfrac{(1/2)^2}{3(3-1/2)}$ = 1.33 minutes

The expected waiting time before being served, W_q = $\dfrac{1+3}{2 \times 3} \times \dfrac{1/2}{3(3-1/2)}$ = 2 minutes 40 seconds

The expected time spent in the system, W_s = $\dfrac{2}{45} + \dfrac{1}{3}$ = 22 minutes 40 seconds

The expected numbers of customers in the system, L_s = $\dfrac{1}{2} \times \dfrac{17}{45}$ = 11.33 minutes

As k approaches to infinity, the expressions of L_q, W_q, W_s and L_s are given by

$$L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)}$$

$$W_q = \frac{\lambda}{2\mu(\mu - \lambda)}$$

W_s = W_q + $\dfrac{1}{}$ ----

$$L_s = L_q + \frac{\lambda}{\mu}$$

μ

Example 3

At Indira Gandhi airport, it takes exactly 6 minutes to land an aeroplane, once it is given the signal to land. Although incoming planes have scheduled arrival times, the wide variability in arrival times produces an effect which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of 6 per hour. This produces occasional stack-ups at the airport, which can be dangerous and costly. Under these circumstances, how much time will a pilot expect to spend circling the field waiting to land?

Solution:

Here, service time is fixed being equal to 6 minutes.
The service distribution is last member of Erlang family, i.e., for $k = \propto$
Mean arrival rate of aeroplanes, λ = 6 per hour
Mean landing rate of planes, $\mu = (1/6) \times 60 = 10$ per hour

$$W_q = \frac{6}{2 \times 10 \times (10 - 6)} = 3/40 \text{ hours} = 4.5 \text{ minutes}$$

Self Test Questions

Theory

1. What do you understand by a queue? Give some important applications of queuing theory.

2. Explain the basic queuing process.

3. What do you understand by queue discipline and input process

4. Explain the constituents of a queuing model.

5. State some of the important distributions of arrival interval and service times.

6. Give the essential characteristics of the queuing process.

Practical

1. Students arrive at the head office of www.universalteacher.com according to a Poisson input process with a mean rate of 30 per hour. The time required to serve a student has an exponential distribution with a mean of 40 per hour. Assume that the students are served by a single individual, find the average waiting time of a student.

2. A barber with a one-man shop takes exactly 30 minutes to complete one haircut. If customers arrive according to a Poisson process at a rate of one every 40 minutes, how long on the average must a customer wait for service?

3. In PNB there is only one window, a solitary employee performs all the service required and the window remains continuously open from 10.00 A.M. to 5.00 P.M. It has been discovered that the average number of clients are 54 during the day and that the average services time is of five minutes per person. Calculate:

i. the average number of clients in the system
ii. the average number of client in the waiting line
iii. the average waiting time,

4. At the PVR Cinema Hall, customers arrive to purchase tickets according to a Poisson process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the value of P_o, L_s and W_s.

5. Patients arrive at the Lifeline Hospital according to a Poisson distribution at the rate of 35 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 25 per hour.

i. Find the effective arrival rate at the clinic.
ii. What is the probability that an arriving patient will not wait?
iii. What is the expected waiting time until a patient is discharged from the clinic?

6. A booking counter at New Delhi Railway Station takes 10 minutes to book a ticket for each customer. If the customers are arriving according to a Poisson process with a rate of 4 per hour, find out

a. Expected queue length.
b. Expected waiting time of a customer in the queue.
c. Expected time a customer spends in the system.

7. Universal Bank is considering opening a drive in window for customer service. Management estimates that customers will arrive at the rate of 12 per hour. The teller

whom it is considering to staff the window can service customers at the rate of one every three minutes.

Assuming Poisson arrivals and exponential service find

1. Average number in the waiting line.
2. Average number in the system.
3. Average waiting time in line.
4. Average waiting time in the system.

8. Himachal fertilizers Ltd. distributes its products' by trucks loaded at its loading station. Both company trucks and contractors' trucks are used for this purpose. Trucks arrive at a rate of 10 per minute and the average loading time is 6 minutes.
You are required to determine

i. The probability that a truck has to wait.
ii. The waiting time of a truck that waits.

9. The Janta transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 6 per hour and the clerk can service 10 customers on the average per hour. You are required to answer the following.

i. What is the average number of customers waiting for the service of the clerk?
ii. What is the average time a customer has to wait before getting service?
iii. What is the average waiting time of a customer in the system?

10. The Priya Cinema has recently released a famous movie (Dinosaur Park). Customers arrive at the box office window, being managed by a single individual according to a Poisson input process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the average waiting time of a customer.

Chapter - 11

Inventory Control Models

Introduction

The word 'inventory' means simply a stock of idle resources of any kind having an economic value. In other words, inventory means a physical stock of goods, which is kept in hand for smooth and efficient running of future affairs of an organization. It may consist of raw materials, work-in-progress, spare parts/consumables, finished goods, human resources such as unutilized labour, financial resources such as working capital, etc. It is not necessary that an organization has all these inventory classes. But whatever may be the inventory items, they need efficient management as generally, a substantial amount of money is invested in them. The basic inventory decisions include:

- How much to order?
- When to order?
- How much safety stock should be kept?

The problems faced by different organizations have necessitated the use of scientific techniques in the management of inventories known as inventory control.

Definition

Inventory control is the technique of maintaining stock items at desired levels. In other words, it is concerned with the acquisition, storage, and handling of inventories so that the inventory is available whenever needed.

What are the factors that affect inventory level?

Inventory models can be classified according to the following factors:

1. **Inventory related costs**

   Inventory related costs are classified as

   - **Purchase (or production) cost.** It is the cost at which an item is purchased, or if an item is produced, it is the direct manufacturing cost. In many practical situations, the unit purchase price depends on the quantity purchased so the purchase price is of special interest when large quantities are bought or when large production runs may result in a decrease in the production cost.

- **Ordering (or replenishment or set up) cost.** The cost incurred in replenishing the inventory is known as ordering cost. It includes all the costs relating to administration (such as salaries of the persons working for purchasing, telephone calls, computer costs, postage, etc.), transportation, receiving and inspection of goods, processing payments, etc. If a firm produces its own goods instead of purchasing the same from an outside source, then it is the cost of resetting the equipment for production. This cost is expressed as the cost per order or per set up. It is denoted by C_o.
- **Carrying (or holding) cost.** The cost associated with maintaining the inventory level is known as holding cost. It is directly proportional to the quantity to be kept in stock and the time for which an item is held in stock. It includes handling cost, maintenance cost, depreciation, insurance, warehouse rent, taxes, etc.
  This cost may be expressed either as per unit of item held per unit of time or as a percentage of average rupee value of inventory held. It is denoted by C_h.
- **Shortage (or stock out) cost.** It is the cost, which arises due to running out of stock (i.e., when an item can not be supplied on the customer's demand). It includes the cost of production stoppage, loss of goodwill, loss of profitability, special orders at higher price, overtime/idle time payments, expediting, loss of opportunity to sell, etc. It is denoted by C_s.

2. Demand

It is an effective desire which is related with a particular time, price, and quantity. The demand pattern of a commodity may be either deterministic or probabilistic. In case of deterministic, it is assumed that the quantities needed in future are known with certainty. This can be fixed (static) or can vary (dynamic) from time to time. To the contrary, in case of probabilistic, the demand over a certain period of time is uncertain, but its pattern can be described by a known probability distribution.

3. Ordering cycle

An ordering cycle is defined as the time period between two successive placement of orders. The order may be placed on the basis of following two types of inventory review systems:

- Continuous review: In this case, record of the inventory level is updated continuously until a specified point (known as reorder point) is reached, at this point a new order is placed. Sometimes, this is referred to as the two-bin system. The inventory is divided into two parts (two bins). Initially, items are used only from one bin, and when it becomes empty, a new order is placed. Demand is then satisfied from the second bin until the order is received. After receiving the order, the second bin is filled to make up the earlier total. The remaining items are placed in the first bin.
- Periodic review: In this case, the orders are placed at equally spaced intervals of time. The quantity ordered each time depends on the available inventory level at the time of review.

4. Time horizon

This is also known as planning period over which the inventory level is to be controlled. This can be finite or infinite depending on the nature of demand.

5. Lead time or delivery lag

The time gap between the moment of placing an order and actually receiving the order is referred to as lead time. The lead time can be deterministic, constant or variable, or probabilistic. If there is no such gap, then we say that lead time is zero. If the lead time exists (i.e., it is not zero), then it is required to place an order in advance by an amount of time equal to the lead time.

6. Buffer (or safety) stock

Normally, demand and lead time are uncertain and cannot be predetermined completely. So to absorb the variation in demand and supply, some extra stock is kept. This extra stock is known as buffer stock.

7. Number of items

Generally, an inventory system involves more than one commodity. The number of items held in inventory affect the situation when these items compete for limited floor space or limited total capital.

8. Government's policy

For items to be imported as well as for other items like explosive, highly inflammable, and other essential items, the Government has laid down some policy norms. All these affect the level of inventories in an organization.

Objectives of Inventory Control

What are the objectives of inventory control?

- To minimize the possibility of delays in production through regular supply of raw materials.
- To keep inactive, waste, surplus, scrap and obsolete items at the minimum level.
- To maintain the overall investment in inventory at the lowest level, consistent with operating requirements.
- To exercise economies in ordering, obtaining, and storing of the materials.

What are the benefits of inventory control?

- It enables the material to be procured in economic quantities.
- It eliminates delays in production caused by the non-availability of required materials.
- It works as a check on the over accumulation of inventories and thereby results in minimum investment consistent with production requirements.
- It reduces inventory losses caused by inadequate inspection of incoming materials and losses due to obsolescence, deterioration, waste and theft while in storage.
- It ensures proper execution of policies covering procurement and use of materials. It also facilitates timely adjustment with changing conditions in the market.

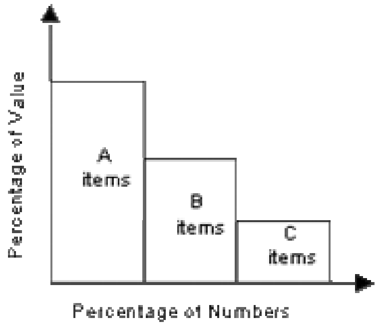
Techniques of Inventory Control

- ABC analysis
- VED analysis

ABC analysis

Under this technique, the inventory items are divided into three groups, viz., A, B and C on the basis of the investment involved.

Category (or group)	Percentage of the items	Percentages of the total annual value of the inventories (Rs.)
A	10-20	70-85
B	20-30	10-25
C	60-70	5-15



ABC classification

In fact, ABC analysis indicates the items of raw materials to be controlled by managers at different levels. The managers are responsible for ensuring optimal investment in raw materials.

VED analysis

This analysis consists of separating the inventory items into three groups according to their critically as under:

1. Vital items (or V items) – These items are considered vital for smooth running of the system and without these items the whole system becomes inoperative. Thus, close attention is paid to V items.
2. Essential items (or E items) – These items are considered essential for efficient running of the system.
3. Desirable items (or D items) – The availability of these items help in increasing the efficiency.

The criticality of the item may either be on technical grounds or on environmental grounds or on both.

ABC analysis coupled with VED analysis enhances the efficiency of control on inventories.

The Basic Deterministic Inventory Models

Before examining the solution of specific inventory models, we provide the notations used in the development of these models.

Q = Number of units ordered per order.
D = Rate of demand.
N = Number of orders placed per year.
TC = Total inventory cost
C_o = Cost of ordering per order
C = Purchase or manufacturing price per unit
C_h = Cost of holding stock per unit per period of time.
C_s = Shortage cost per unit.
R = Reorder point
L = Lead time (weeks or months)
t = The elapsed time between placement of two successive orders.

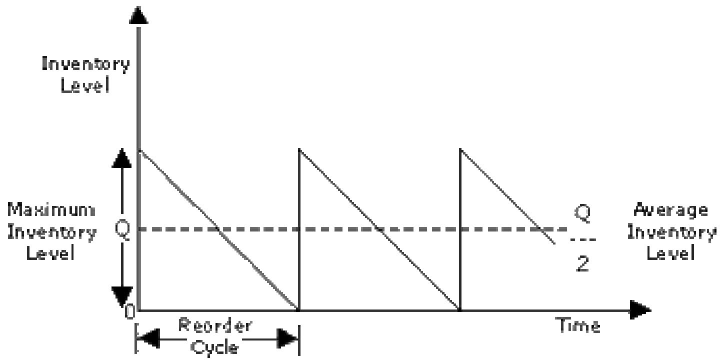
Model I - Economic Order Quantity Model With Uniform Demand

The main problem while purchasing material is how much to buy at a time. If large quantities are bought, the cost of carrying the inventory would be high. To the contrary, if frequent purchases are made in small quantities, costs relating to ordering will be high. So the problem is of indecision. How this problem can be resolved?

EOQ is that size of the order for which the cost of maintaining inventories is minimum. Therefore, the quantity to be ordered at a given time should be determined by taking into account two factors, i.e., the acquisition cost and the cost of possessing materials. We illustrate this model after making the following assumptions:

Assumptions

- Demand rate is uniform over time and is known with certainty.
- The inventory is replenished as soon as the level of the inventory reaches to zero. Thus shortages are not allowed.
- Lead time is zero.
- The rate of inventory replenishment is instantaneous.
- Quantity discounts are not allowed.



The inventory costs are determined as follows:

1. Ordering cost = Number of orders per year X Ordering cost per order
= N X C_o
= Total annual demand/Number of units ordered X C_o

$$= \frac{D}{Q} \times C_o$$

2. Carrying cost = Average inventory X carrying cost per unit

$$= \frac{Q}{2} \times C_h$$

The total inventory cost is the sum of ordering cost and carrying cost.

$$TC = \frac{D}{Q} C_o + \frac{Q}{2} C_h$$

The total cost is minimum at a point where ordering cost equals carrying cost. Thus, economic order quantity occurs at a point where
Ordering cost = Carrying cost

$$\frac{D}{Q} C_o = \frac{Q}{2} C_h$$

Thus, optimal Q* (EOQ) is derived to be

EOQ = $Q^* = \sqrt{\dfrac{2DC_o}{C_h}}$

The period t is given by $t^* = \dfrac{Q^*}{D} = \sqrt{\dfrac{2C_o}{C_hD}}$

Optimal number of orders per year is given by $N^* = \dfrac{D}{Q^*} = \dfrac{1}{t^*}$

Minimum total yearly inventory cost $TC^* = \sqrt{2DC_oC_h}$

Example 1

Annual usage	500 pieces
Cost per piece	Rs. 100
Ordering cost	Rs. 10 per order
Inventory holding cost	20% of Average Inventory

Solution:

D = 500 pieces
C_o = 10
C_h = 100 X 20% = Rs. 20

EOQ = $\sqrt{\dfrac{(2 \text{ X } 10 \text{ X } 500)}{20}}$

EOQ = 22 pieces (rounded)

Example 2

The Newtech Hardware Company sells hardware items. Consider the following information.

Annual sales = Rs. 10000
Ordering cost = Rs. 25 per order

Carrying cost = 12.5% of average inventory value.
Find the optimal order size, number of orders per year, and cycle period.

Solution:

D = Rs. 10000, C_o = Rs. 25, C_h = 12.5% of average inventory value/unit

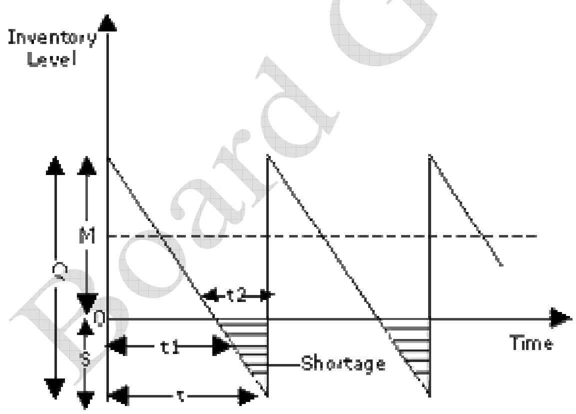
$$\text{EOQ} = Q^* = \sqrt{\frac{(2 \times 25 \times 10000)}{(0.125)}} = \text{Rs. } 2000$$

$$t^* = \frac{Q^*}{D} = \frac{2000}{10000} = 1/5 \text{ yrs} = 73 \text{ days}$$

$$N^* = \frac{1}{t^*} = \frac{1}{1/5} = 5$$

Model II - EOQ When Shortages Are Allowed

In this case, shortages are permitted which implies that shortage cost is finite or it is not large. The cost of a shortage is assumed to be directly proportional to the mean number of units short. Further, all the assumptions of model I hold good here also. The model is graphed in the following figure.



Where
S = Back order quantity.
M = Maximum inventory level.
t1 = Time during which stock is available.

t2 = Time during which there is a shortage.
t = Time between receipt of orders.

$$Q^* = \sqrt{\left[\frac{2DC_o}{C_h} \times \frac{(C_h + C_o)}{C_s}\right]}$$

$$M^* = \sqrt{\left[\frac{(2DC_o)}{(C_h)} \times \frac{C_s}{(C_h + C_s)}\right]}$$

$$t^* = \sqrt{\left[\frac{2C_o}{DC_h} \times \frac{(C_s + C_h)}{C_s}\right]}$$

$$T^{C*} = \sqrt{\left[2DC_o C_h \times \frac{C_s}{(C_s + C_h)}\right]}$$

Example

The Wartsila Diesel Company has to supply diesel engines to a truck manufacturer at a rate of 10 engines per day. The ordering cost is Rs. 150 per order. The penalty in the contract is Rs. 90 per engine per day late for missing the scheduled delivery date. The cost of holding an engine in stock for one month is Rs. 140. His production process is such that each month (30 days) he starts procuring a batch of engines through the agencies and all are available for supply after the end of the month. Determine the maximum inventory level at the beginning of each month.

Solution:

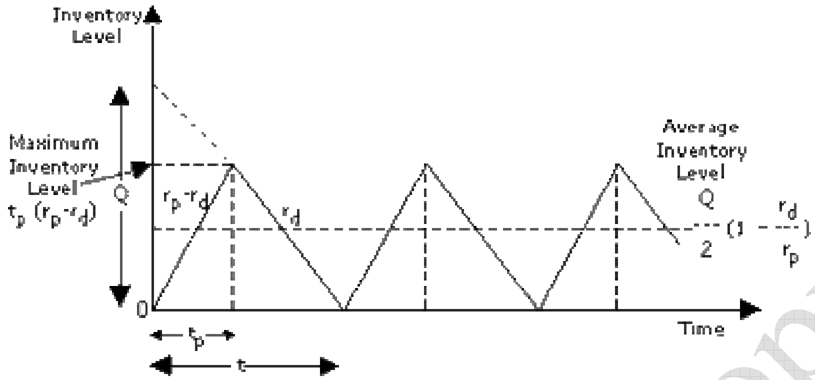
Given
Demand (D) = 10 engines per day
Shortage cost (C_s) = Rs. 90 per day per engine
Carrying cost (C_h) = 140/30 = 14/3 per engine per day
Ordering cost (C_o) = Rs. 150 per order

$$M^* = \sqrt{\left[\frac{2 \times 10 \times 150}{14/3} \times \frac{90}{14/3 + 90}\right]} \times 30$$

= 741.65 = 742 engines (approx.)

Model III - EOQ With Uniform Replenishment

In the previous models, we assumed that the replenishment time is zero, i.e., the entire order was received just as inventory ran out of stock. However, in real life situations, the production run may take a significant time to complete.



$$Q^* = \sqrt{\frac{(2DC_o)}{C_h} \cdot \frac{r_p}{(r_p - r_d)}}$$

$$N^* = \sqrt{\frac{DC_h (r_p - r_d)}{2C_o r_p}}$$

$$t_p^* = \sqrt{\frac{2DC_o}{2C_h r_p (r_p - r_d)}}$$

$$TC^* = \sqrt{\left[2DC_oC_h \right] \left[1 - \frac{r_d}{r_p} \right]}$$

Where:
r_p = production rate in units per time period
r_d = demand rate in units per time period
t_p = length of production run

Example

The Long Ride Tyre Company produces 600 tyres per day and sells them at approximately half that rate. Accounting figures show that the production set-up cost is

Rs. 500 and carrying cost per unit is Rs. 2.5. If annual demand is 60000 tyres, what is the optimal lot size and how many production runs should be scheduled per year?

Solution:

Given
Annual demand (D) = 60000 tyres
Carrying cost (C_h) = Rs. 2.5
Setup cost (C_s) = Rs. 500
Production rate (r_p) = 600 tyres
Demand rate (r_d) = 300 tyres

$$Q^* = \sqrt{\frac{(2 \times 60000 \times 500 \times 600)}{2.50 \times (600 - 300)}}$$

= 6929 tyres.

$$N^* = \frac{60000}{6929}$$

= 8.65 runs/year or 9 runs/year (approx.)

Model IV - EOQ With Quantity Discounts

a) Inventory Model With Single Discount

The purchase inventory model with single discount may be expressed as follows:

Order Quantity	Unit Price (Rs.)
$1 \leq Q_1 < b$	P_1
$b \leq Q_2$	P_2

Following are the steps to summarize the approach.

Steps

1. Compute the optimal order quantity for the lowest price (highest discount), i.e.,

$$Q_2^* = \sqrt{\frac{(2DC_o)}{C_hP_2}}$$

and compare the value of Q_2* with the quantity b which is required to avail the discount. If $Q_2^* \geq b$, then place orders for quantities of size Q_2* and obtain discount; otherwise move to step 2.

2. Compute Q_1* for price P_1 and compare $TC(Q_1^*)$ with TC(b). The values of $TC(Q_1^*)$ and TC(b) may be determined as follows:

$$TC(Q_1^*) = DP_1 + (D/Q_1^*) \times C_o + (Q_1^*/2) \times C_h \times P_1$$

$$TC(b) = DP_2 + (D/b) \times C_o + (b/2) \times C_h \times P_2$$

If $TC(Q_1^*) > TC(b)$, then place orders for quantities of size b to get the discount.

Example

A big cold drinks company, the Piyo - Pilao Company, buys a large number of pallets every year, which it uses in the warehousing of its bottled products. A local vender has offered the following discount schedule for pallets:

Order Quantity	Unit Price (Rs.)
Upto 699	10.00
700 and above	9.00

The average yearly replacement is 2000 pallets. The carrying costs are 12% of the average inventory and ordering cost per order is Rs. 100.

Solution:

Given
D = 2000 pallets/year, $C_h = 0.12$, C_o = Rs. 100, P_1 = Rs. 10, P_2 = Rs. 9.00

Step 1

The lowest price (highest discount) is RS. 9.00.

$$Q_2^* = \sqrt{\frac{(2 \text{ X } 2000 \text{ X } 100)}{0.12 \text{ X } 9}}$$

= 608.58 pallets/order

Since $Q_2^* < b$ (i.e., 608 < 700), Q_2^* is not feasible.

Step 2

$$Q_1^* = \sqrt{\frac{(2 \text{ X } 2000 \text{ X } 100)}{0.12 \text{ X } 10}}$$

= 577.35 pallets/order

$TC(Q_1^*)$ = TC(577.35) = 2000 X10 + (2000/577.35) X 100 + (577.35/2) X 0.12 X 10 = Rs. 20692.82

TC(b) = TC (700) = 2000 X 9 + (2000/700) X 100 + (700/2) X 0.12 X 9 = Rs. 18663.71

Since TC(b) < $TC(Q_1^*)$ and hence the optimal order quantity is the price discount quantity, i.e., 700 units.

b) Inventory model with double discount

Order Quantity	Unit Price (Rs.)
$1 \le Q_1 < b_1$	P_1
$b_1 \le Q_2 < b_2$	P_2
$b_2 \le Q_3$	P_3

Where b_1 and b_2 are the quantities, which determine the price discount?

Following are the steps to summarize the approach.

Steps

1. Compute the optimal order quantity for the lowest price (highest discount), i.e., Q_3^* and compare it with b_2

   a. If $Q_3^* \ge b_2$, then place order equal to this optimal quantity Q_3^*
   b. If $Q_3^* < b_2$, then go to step 2

2. Compute Q_2^* and since $Q_3^* < b_2$, this implies Q_2^* is also less than b_2. Thus, either $Q_2^* < b_1$ or $b_1 \le Q_2^* < b_2$

a. If $Q_2^* < b_2$, but $\ge b_1$, then proceed as in the case of single discount, i.e., compare $TC(Q_2^*)$ and $TC(b_2)$ to determine the optimal purchase quantity.
b. If $Q_2^* < b_2$ and b_1, then move to step 3

3. Compute Q_1^* and compare $TC(b_1)$, $TC(b_2)$ and $TC(Q_1^*)$ to determine the purchase quantity.

Example

A large dairy firm, the Cow and Buffalo Company, buys bins every year, which it uses in the warehousing of its bottled products. A local vender has offered the following discount schedule for bins:

Order Quantity	Unit Price (Rs.)
Upto 699	10.00
700 to 949	9
950 and above	8

The average yearly replacement is 2000 bins. The carrying costs are 12% of the average inventory and ordering cost per order is Rs. 100.

Solution:

Given
$D = 2000$ bins/year, $C_h = 0.12$, $C_o =$ Rs. 100, $P_1 =$ Rs. 10, $P_2 =$ Rs. 9, $P_3 =$ Rs. 8

Step 1

The lowest price (highest discount) is Rs. 8. Thus calculating Q_3^* = corresponding to this range as follows:

$$Q_3^* = \sqrt{\frac{(2 \text{ X } 2000 \text{ X } 100)}{0.12 \text{ X } 8}}$$

$$= 645.49 \text{ bins/order}$$

Since $Q_3^* < b_2$ (i.e., 645.49 < 950), go to step 2 to determine Q_2^*

Step 2

$$Q_2^* = \sqrt{\frac{(2 \text{ X } 2000 \text{ X } 100)}{0.12 \text{ X } 9}}$$

= 608.58 bins/order

Again, since $Q_2^* < b_2$ and b_1 (i.e., 608.58 < 950 & 700) go to step 3 to calculate Q_1^* and compare total inventory cost corresponding to Q_1^*, b_1 and b_2.

Step 3

$$Q_1^* = \sqrt{\frac{(2 \text{ X } 2000 \text{ X } 100)}{0.12 \text{ X } 10}}$$

= 577.35 bins/order

$TC(Q_1^*)$ = TC(577.35) = 2000 X10 + (2000/577.35) X 100 + (577.35/2) X 0.12 X 10
= Rs. 20692.82

$TC(b_1)$ = TC(700) = 2000 X 9 + (2000/700) X 100 + (700/2) X 0.12 X 9
= Rs. 18663.71

$TC(b_2)$ = TC(950) = 2000 X 8 + (2000/950) X 100 + (950/2) X 0.12 X 8
= Rs. 16666.52

The lowest total inventory cost is $TC(b_2)$ = Rs. 16666.52 and hence the optimal order quantity is the price discount quantity of 950 units, i.e., $Q^* = b_2$ = 950 units.

Probabilistic or Stochastic Models

The previous sections have assumed that the data required by a model are known exactly. But in actual business life, you never know all the values with perfect certainty.

Single Period Discrete Probabilistic Demand Model

For a given item, the following factors are involved in the determination of C_1 and C2:

i. Unit selling price (S)
ii. Unit purchase cost (C)
iii. Carrying cost for the entire period (Ch)
iv. Salvage value (V)
v. Shortage penalty cost (Cs)

The unit costs of over-ordering and under-stocking are then

C1 = C + Ch - V
C2 = S - C + Ch/2 + Cs

Example

A trader stocks a particular seasonal product at the beginning of the season and cannot reorder. The item costs him Rs. 25 each and he sells at Rs. 50 each. For any item that cannot be met on demand, the trader has estimated a goodwill cost of Rs. 15. Any item unsold will have a salvage value of Rs. 10. Holding cost during the period is estimated to be 10 percent of the price. The probability distribution of demand is given below.

Units stocked	2	3	4	5	6
Probability of demand, p(D=Q)	0.35	0.25	0.20	0.15	0.05

Determine the optimal number of items to be stocked.

Solution:

Given
S = Rs. 50, C = Rs. 25, C_h = 0.10 X 25 = 2.5, V = Rs. 10, C_s = 15.
The probability distribution of demand is given in the following table.

Units stocked	**Probability of Demand p (D=Q)**	**Cumulative probability P (D≤Q)**
2	0.35	0.35
3	0.25	0.60
4	0.20	0.80
5	0.15	0.95
6	0.05	1.00

C_1 = 25 + 2.5 - 10 = 17.5 , C_2 = 50 - 25 - (2.5/2) + 15 = 38.75
the ratio,

$$\frac{C_2}{C_1 + C_2} = \frac{38.75}{17.5 + 38.75} = 0.69$$

In the above table, the ratio (0.69) lies between cumulative probabilities of 0.60 and 0.80, which in turn reflect the values of Q as 3 and 4. That is,
$P(D \leq 3) = 0.60 < 0.69 < 0.80 = P(D \leq 4)$.

Therefore, the optimal number of units to stock is 4 units.

Inventory for Perishable Products

Example

A newspaper boy buys papers for Rs. 0.35 each and sells them for Rs. 0.60 each. He can't return unsold newspapers. Daily demand has the following distribution:

No. of customers	230	240	250	260	270	280	290	300	310	320
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

If each day's demand is independent of the previous day's demand, how many papers should he order each day?

Solution:

The probability distribution of demand is given in the following table.

No. of customers	230	240	250	260	270	280	290	300	310	320
Probability	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05
Cumulative Probability	0.01	0.04	0.10	0.20	0.40	0.65	0.80	0.90	0.95	1.00

$C_1 = 0.35 + 0 - 0 = 0.35$
$C_2 = 0.60 - 0.35 - (0/2) + 0 = 0.25$

$$\frac{C_2}{C_1 + C_2} = \frac{0.25}{0.35 + 0.25} = 0.416$$

From the above table, we notice that the computed value of 0.416 lies between 0.40 and 0.65 corresponding to 270 and 280 customers respectively. Hence 280 being the higher value is the optimal no. of papers to be stocked by the newspaper boy.

Self Test Questions

Theory

1. What are inventories?

2. What are the objectives that should be fulfilled by an inventory control system?

3. Explain the factors that affect the inventory level in an organization.

Practical

1. An item is produced at the rate of 50 units per day and is consumed at the rate of 25 units per day. If the set up cost is Rs. 100 per production run and holding cost in stock is Rs. 365 per unit per year, find

i. economic lot size per run
ii. number of runs per year
iii. total related cost

2. An item is required at a rate of 18000 units per year. Storage cost is Rs. 0.10 per unit per month. If the cost of placing an order is Rs. 400, find

i. EOQ
ii. number of orders per year
iii. cycle period
iv. total annual cost if per unit cost is Rs.2.

3. A company uses annually 48000 units of a raw material costing Rs. 1.2 per unit. Placing each order cost Rs. 45 and the carrying cost is 15% per year of the average inventory.

i. Find the economic order quantity.
ii. Supposing that the company follows the EOQ purchasing policy that it operates for 300 days a year, that the procurement time is 12 days and the safety stock is 500 units, find the reorder point, the maximum, minimum and average inventories.

4. A manufacture has to supply his customer 24000 units of his product. The demand is fixed and known. The customer has no storage space and so why the manufacturer has to ship a day's supply each day. If the manufacturer fails to supply, the penalty is Rs. 0.20 per month. The inventory holding cost is Rs. 0.10/unit/month and the set up cost is Rs. 350 per run. Find the optimal lot size for the manufacturer.

5. A manufacturer has to supply his customer with 600 units of his product per year. Shortages are not allowed and the storage cost amount to Rs. 0.60 per unit/year. The set up cost per run is Rs. 80.00. Find the optimal run size and the minimum average yearly cost.

Chapter – 12

Dynamic Programming

Introduction

The term 'dynamic programming' refers to a general optimization technique useful for solving a class of multistage problems.

Definition

Dynamic programming is a methodology useful for solving problems that involve taking decisions over several stages in a sequence.

For instance, consider a company that has to decide on the production plan of' an item for the next three months, so as to meet the demands in different months at minimum cost. The different months for which the production is to be decided, constitute the stages. So it is a multistage problem. These type of problems and a variety of other business problems such as inventory control, replacement, scheduling, capital budgeting, etc., where decisions are made sequentially over several periods can be solved by dynamic programming approach. One thing common to all models in this category is that current decisions influence both present & future periods.

The dynamic programming approach divides the problem into several sub-problems or stages and then these sub-problems are solved sequentially until the initial problem is finally solved. The common characteristic of all dynamic models is expressing the decision problem by means of a recursive formulation (recursion means determination of a successive element by operations on a preceding element according to a formula).

Basic Terminology

Stage

The point at which a decision is made is known as a stage. The end of a stage marks the beginning of the immediate succeeding stage. For instance, in the salesmen allocation problem, each territory represents a stage; in the shortest route problem, each city represents a stage.

State

Conceptually, the variable that links two stages in a multistage decision problem is called a state variable. At any stage, the values that state variables can take describe the status of the problem. These values are referred to as states. For example, in the shortest route problem, a city is referred to as state variable.

Principle of Optimality

The principle of optimality states that the optimal decision from any state in a stage to the end is independent of how one actually arrives at that state.

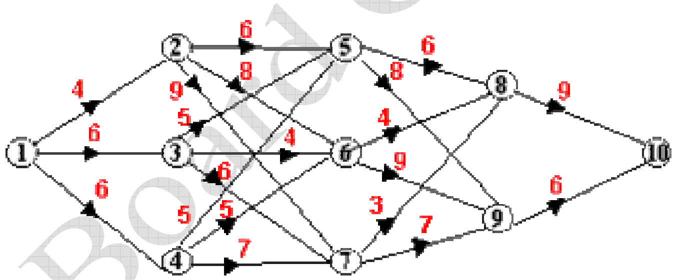
Optimal Policy

A policy which optimizes the value of an objective function is called an optimal policy.

The Shortest Route Problem

Example

The MD of www.universalteacher.com wants to visit the Bell Well temple. Consider the following diagram where circles denote states, and lines between two such circles represent highways connecting the states. The numbers inside the circles represent state numbers, and those given beside the lines denote the distances between the states connected by the lines. The problem is to find the shortest route from state 1 to state 10 where the Bell Well temple is situated.



Solution:

For a systematic approach to dynamic programming problem, consider the following notations.

n = number of stages.
s = state variable.

d_{sj} = immediate distance from entering state s to existing state j.
$f_n(s)$ = the overall optimal objective function with n more stages to go when he is in state s.
$j_n(s)$ = a decision yielding $f_n(s)$.

Notice that the entire trip from state 1 to state 10 requires four stages (highways), regardless of the particular routing. Now, the problem is to select these four highways so that the total distance covered is least. The first highway has to be chosen from 1-2, 1-3, or 1-4, as 1 is the starting state. Likewise, the second highway has to be chosen from 2, 3, or 4, the third from 5, 6, or 7 and the fourth from 8 or 9.

There is one table for each possible stage n, namely, n = 1, 2, 3, and 4. We start calculating distances between pair of states from stage 4 backwards. At the beginning of stage 4, we can be in either 8 or 9 (states). We note that state 10 is the only destination from both states 8 & 9. We summarize this information in the format below.

Stage 4 (n = 1)

State	Decision (j)	Best decision	Best distance
(s)	10	$j_1(s)$	$f_1(s)$
8	9 + 0	10	9
9	6 + 0	10	6

Stage 3 (n = 2)

State	Decision (j)		Best Decision	Best Distance
(s)	8	9	$j_2(s)$	$f_2(s)$
5	6 + 9 = 15	8 + 6 = 14	9	14
6	4 + 9 = 13	9 + 6 = 15	8	13
7	3 + 9 = 12	7 + 6 = 13	8	12

The entries in second & third column are the sum of the immediate distance dsj to go from state s to state j. In each row, we examine the sums to find the smallest. Observe that $f_1(8) = 9$ is added to each d_{s8} in the j = 8 column and $f_1(9) = 6$ is added to each d_{s9} in the j = 9 column. The above table shows that with two stages left it is optimal to go to state 9 from state 5, and state 8 from states 6 & 7.

The analysis for n = 3 appears in the following table.

Stage 2 (n = 3)

State	Decision (j)			Best Decision	Best Distance
(s)	5	6	7	$j_3(s)$	$f_3(s)$
2	6 + 14 = 20	8 + 13 = 21	9 + 12 = 21	5	20
3	5 + 14 = 19	4 + 13 = 17	6 + 12 = 18	6	17
4	5 + 14 = 19	5 + 13 = 18	7 + 12 = 19	6	18

Stage 1 (n = 4)

State	Decision (j)			Best Decision	Best Distance
(s)	2	3	4	$j_4(s)$	$f_4(s)$
1	4 + 20= 24	6 + 17 = 23	6 + 18 = 24	3	23

The computations terminate in the above table with n = 4. The shortest route from state 1 to state 10 is given by 1-3-6-8-10, and the distance to be covered is 23.

Inventory Control

Example

Spitzen Ltd. has to supply the following number of items at the end of each month.

Month No.	Month	No. of items
1	January	100
2	February	200
3	March	300
4	April	400
	Total	1000

Production during a month is available for supply at the end of the month. The stock holding cost per month is Re. 1 per item. The setup cost is Rs. 900 per setup and Rs. 2 per item. Find the optimal policy so that total cost may be minimum.

Solution:

The production cost Rs. 2 per item is always incurred, whether items are produced in the beginning or at any other time.
Fixed cost = 1000 X 2 = Rs. 2000

Case I - April

Requirement = 400
Cost = Rs. 900 (setup cost)

Case II - March

We have the following alternatives.

1. Produce 700 (demand for March & April) items in the beginning of March.
Cost = 900 + (400 X 1) = Rs. 1300

2. Produce 300 items in March & 400 in April.
Cost = 900 + 900 = Rs. 1800

Hence, the optimal sub-policy for March is: Produce 700 items in March.

Case III - February

We have the following alternatives.

1. Produce 900 items in the beginning of February.
Cost = 900 + 700 X 1 + 400 X 1 = Rs. 2000

2. Produce 500 items now & 400 in April.
Cost = 900 + 300 X 1 + 900 = Rs. 2100

3. Produce 200 items now & 700 in March.
Cost = 900 + 1300 = Rs. 2200

The optimal sub-policy for March was to produce 700 items. Therefore, in February we have not considered the following case: Produce 200 items now, 300 items in March & 400 in April.
Hence, the optimal sub-policy for February is: Produce 900 items in February.

Case IV - January

We have the following alternatives.

1. Produce 1000 items in the starting of January.
Cost = 900 + (900 X 1 + 700 X 1 + 400 X 1) = Rs. 2900

2. Produce 600 items now & 400 in April.
Cost = 900 + 500 X 1 + 300 X 1+ 900 = Rs. 2600

3. Produce 300 items now & 700 in March.
Cost = 900 + 200 X 1 + 1300 = Rs. 2400

4. Produce 100 items now & 900 in February.
Cost = 900 + 2000 = Rs. 2900

The minimum cost is Rs. 2400. Hence, the best policy is: Produce 300 items in January & 700 in March.

Solution Of LP By DP

Example 1

Maximize $z = 5x_1 + 9x_2$

Subject to

$-x_1 + 5x_2 \leq 3$
$5x_1 + 3x_2 \leq 27$

$x_1, x_2 \geq 0$

Solution:

Given
$-x_1 + 5x_2 \leq 3$ ....(i)
$5x_1 + 3x_2 \leq 27$ ....(ii)

Let R_1 & R_2 be the resources associated with first and second constraint respectively. The maximum value of the resources are specified in the RHS of the two constraints, i.e., $R_1 = 3$ & $R_2 = 27$.
From equation (i), if we are deciding only on x_2 and RHS is R1, then $5x_2$ has to be less than or equal to R_1, i.e., $x_2 \leq R_1/5$.

Similarly, from equation (ii), we have
$X2 \leq R_2/3$.

Since we are maximizing, the maximum value of x_2 has to be equal to the minimum of $R_1/5$ and $R_2/3$.

$f_2(R_1, R_2) = \text{Max}\ (9x_2)$
$= 9\ \text{Max}\ (x_2)$
$= 9\ \text{Min}\ (R_1/5, R_2/3)$ -------(iii)

$f_1(3, 27) = \text{Max}\ [5x_1 + f_2(3 + x_1, 27 - 5x_1)]$ ------(iv)
$0 \le x_1 \le 27/5$

From equation (iii),
$f_2(R_1,R_2) = 9\ \text{Min}\ (R_1/5, R_2/3)$ ----(iv)

Therefore, $f_2(3 + x_1, 27 - 5x_1) = 9\ \text{Min}\ [(3 + x_1)/5, (27 - 5x_1)/ 3]$

From equation (iv)
$f_1(3, 27) = \text{Max}\ \{5x_1 + 9\ \text{Min}[\ (3 + x_1)/5, (27 - 5x_1)/3]\ \}$ -----(v)

We now find the range of x_1 for which $(3 + x_1)/5 < (27 - 5x_1)/3$.

Comparing $(3 + x_1)/5$ & $(27 - 5x_1)/3$, we get

$$\frac{(3 + x_1)}{5} = \frac{(27 - 5x_1)}{3}$$

$3(3 + x_1) = 5(27 - 5x_1)$
$9 + 3x_1 = 135 - 25x_1$
$\therefore\ x_1 = 4.5$

From equation v), we have
$f_1(3,27) = \text{Max}\ [5x_1 + 9(3 + x_1)/5]$ if $x_1 \le 4.5$
$= \text{Max}\ [5x_1 + 9(27 - 5x_1)/3]$ if $x_1 > 4.5$

From the above, the maximum occurs at $x_1 = 4.5$
$x_2 = \text{Min}\ [3 + 4.5)/5 ,(27 - 5 \text{ X } 4.5)/3]$
$= \text{Min}\ (7.5/5, 4.5/3) = 1.5$

Hence, the optimal solution is

$x_1 = 4.5, x_2 = 1.5$
$z = 5 \text{ X } 4.5 + 9 \text{ X } 1.5 = 22.5 + 13.5 = 36$

Example 2

Maximize $Z = 3x_1 + 5x_2$

Subject to

$x_1 \le 4$
$x_2 \le 6$
$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

Solution:

$x_1 \le 4$ ....(i)
$x_2 \le 6$ ....(ii)
$3x_1 + 2x_2 \le 18$ ....(iii)

Let R_1, R_2 & R_3 be the resources associated with first, second and third constraint respectively.
The maximum value of the resources are specified in the RHS of the two constraints, i.e., $R_1 = 4$, $R_2 = 6$ & $R_3 = 18$.

From equation (ii), if we are deciding only on x2 and RHS is R2 then x_2 has to be less than equal to R_2, i.e., $x_2 \le R_2$.

Similarly, from equation (iii), we have
$2x2 \le R_3$
or $x2 \le R_3/2$

Since we are maximizing, the maximum value of x_2 has to be equal to the minimum of R_2 & $R_3/2$.
$f_2(R_1, R_2, R_3) = \text{Max}\ (5x_2)$
$= 5\ \text{Max}\ (x_2)$
$= 5\ \text{Min}\ (R_2, R_3/2)$ ....(iv)

$f_1(4,6,18) = \text{Max}\ [3x_1 + f2(4 - 2x_1, 6, 18 - 3x_1)]$
$x_2 \le 2,\ x_1 \le 6,\ x_1 \ge 0$

From equation (iv) & (v)

$f1(R_1, R_2, R_3) = \text{Max}\ [3x_1 + 5\ \text{Min}\ (6, (18 - 3x_1)/2]$ ....(vi)

We now find the range of x_1 for which
$6 < (18 - 3x_1)/2$

Comparing 6 & $(18 - 3x_1)/2$, we get
$12 = 18 - 3x_1$
or $3x_1 = 6$
or $x_1 = 2$

From equation (vi), we have:
$f_1(4, 6, 18) = \text{Max } [3x_1 + 5 \text{ X } 6]$ if $x \leq 2$
$= \text{Max } [3x_1 + 5(18 - 3x_1)/2]$ if $x_1 > 2$

From the above, the maximum occurs at $x_1 = 2$.
$x_2 = \text{Min } [6, (18 - 3 \text{ X } 2)/2] = 6$

The values for x_1 and x_2 are 2 and 6. The corresponding value of the objective function is
Z = 3 X 2 + 5 X 6 = 36

An Electronic Device Problem

Example

An electronic device consists of three main components. The failure of one of the components results in the failure of the whole device because the three components are arranged in series. Therefore, it is decided that the reliability (prob. of not failure) of the device can be increased by installing parallel units on each component. Each component may be installed at most 3 parallel units. The total capital (in thousands) available for the device is 10. Consider the following data:

m_i	i = 1		i =2		i = 3	
	r_1m_1	c_1m_1	r_2m_2	c_2m_2	r_3m_3	c_3m_3
1	.5	2	.7	3	.6	1
2	.7	4	.8	5	.8	2
3	.9	5	.9	6	.9	3

Here m_i is the number of parallel units placed with i^{th} component, r_im_i is the reliability of the i^{th} component and c_im_i is the cost for the i^{th} component. Determine m_i which will maximize the total reliability of the system without exceeding the given capital.

Solution:

In this example, we consider each component as one stage. Let the state x_i be defined as the capital allocated stages 1, 2, ..., i. The reliability $r_i m_i$ is a function of $c_i m_i$, i.e., $r_i m_i$ $(c_i m_i)$.

In general the recursive equation is $f_i(x_i)$ = Max. $\{r_i m_i\ (c_i m_i)\ f_{i-1}(x_i - c_i m_i)\}$

where m_i = 1, 2, 3.

$0 \le c_i m_i \le x_i$, i = 1, 2, 3.

There is one table for each possible stage n, namely, n = 1, 2, and 3. We summarize this information in the format below:

Stage 1

State	Evaluations of feasible alternatives $f_1(x_1) = r_1 m_1\ (c_1 m_1)$						Maximum reliability	
	$m_1 = 1$		$m_1 = 2$		$m_1 = 3$			
x_1	$r_1 m_1 = .5$	$c_1 m_1 = 2$	$r_1 m_1 = .7$	$c_1 m_1 = 4$	$r_1 m_1 = .9$	$c_1 m_1 = 5$	$f_1(x_1)$	m_1*
0	-		-		-		-	-
1	-		-		-		-	-
2	.5		-		-		.5	1
3	.5		-		-		.5	1
4	.5		.7		-		.7	2
5	.5		.7		.9		.9	3
6	.5		.7		.9		.9	3
7	.5		.7		.9		.9	3
8	.5		.7		.9		.9	3
9	.5		.7		.9		.9	3
10	.5		.7		.9		.9	3

The analysis for n = 2 appears in the following table.

Stage 2

State	$f_2(x_2) = r_2m_2\ (c_2m_2).\ f_1(x_2 - c_2m_2)$			Maximum reliability	
	$m_2 = 1$	$m_2 = 2$	$m_2 = 3$		
x_2	$r_2m_2 = .7$ $c_2m_2 = 3$	$r_2m_2 = .8$ $c_2m_2 = 5$	$r_2m_2 = .9$ $c_2m_2 = 6$	$f_2(x_2)$	m_2*
0	-	-	-	-	-
1	-	-	-	-	-
2	-	-	-	-	-
3	.7 X (-) = -	-	-	-	-
4	.7 X (-) = -	-	-	-	-
5	.7 X .5 = .35	.8 X (-) = (-)	-	.35	1
6	.7 X .5 = .35	.8 X (-) = (-)	.9 X (-) = (-)	.35	1
7	.7 X .7 = .49	.8 X .5 = .40	.9 X (-) = (-)	.49	1
8	.7 X .9 = .63	.8 X .5 = .40	.9 X .5 = .45	.63	1
9	.7 X .9 = .63	.8 X .7 = .56	.9 X .5 = .45	.63	1
10	.7 X .9 = .63	.8 X .9 = .72	.9 X .7 = .63	.72	2

Note that in case $m_2 = 1$, $f_1(x_2 - c_2m_2)$ has no value until $x_2 - c_2m_2 \leq 1$ or $x_2 \leq 1 + c_2m_2 = 1 + 3 = 4$. Similar is the case with other columns in this table.

Stage 3

State	$f_3(x_3) = r_3m_3\ (c_3m_3).\ f_2(x_3 - c_3m_3)$			Maximum reliability	
	$m_3 = 1$	$m_3 = 2$	$m_3 = 3$		
x_3	$r_3m_3 = .6$ $c_3m_3 = 1$	$r_3m_3 = .8$ $c_3m_3 = 2$	$r_3m_3 = .9$ $c_3m_3 = 3$	$f_3(x_3)$	m_3*
0	-	-	-	-	-
1	.6 X (-) = (-)	-	-	-	-
2	.6 X (-) = (-)	.8 X (-) = (-)	-	-	-
3	.6 X (-) = (-)	.8 X (-) = (-)	.9 X (-) = (-)	-	-
4	.6 X (-) = (-)	.8 X (-) = (-)	.9 X (-) = (-)	-	-
5	.6 X (-) = (-)	.8 X (-) = (-)	.9 X (-) = (-)	-	-
6	.6 X .35 = .210	.8 X (-) = (-)	.9 X (-) = (-)	.210	1
7	.6 X .35 = .210	.8 X .35 = .280	.9 X (-) = (-)	.280	2
8	.6 X .49 = .294	.8 X .35 = .280	.9 X .35 = .315	.315	3
9	.6 X .63 = .378	.8 X .49 = .392	.9 X .35 = .315	.392	2
10	.6 X .63 = .378	.8 X .63 = .504	.9 X .49 = .441	.504	2

The maximum reliability is .504. for the $m_3* = 2$. Also for this, the optimal $f_2(x_3 - c_3m_2) = f_2(8) = .63$. Hence the corresponding $m_2* = 2$. Similarly, the corresponding $m_1* = 3$.

Self Test Questions

Theory

1. What is Dynamic Programming? In what areas of management can it be applied successfully?

2. Discus briefly:

- The general similarities between dynamic and linear programming
- How dynamic programming differs from linear programming?

3. Define the following terms

- Stage
- State
- Principle of optimality

Practical

1. Maximize $z = x_1 + 9x_2$

Subject to
$2x_1 + x_2 \leq 25$
$x_2 \leq 11$

$x_1, x_2 \geq 0$

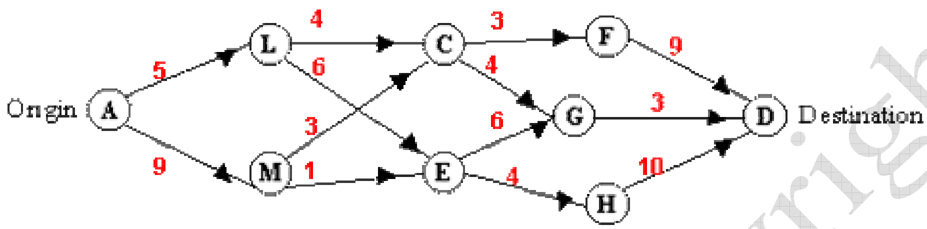
2. Maximize $z = 3x_1 + x_2$

Subject to
$x_1 \leq 2$
$x_2 \leq 4$
$2x_1 + x_2 \leq 6$

$x_1, x_2 \geq 0$

3. The XYZ Trucking Company has to deliver a shipment of goods from city A to city D as shown below. The numbers on the arcs represent the estimated driving times in hours between adjacent cities. The company wants to determine the route requiring the shortest travel time.

Solve the problem and find the minimum time and its associated optimal route.



Chapter – 13

Replacement Models

Introduction

In fact, in any system the efficacy (efficiency) of an item deteriorates with time. In such cases, either the old item should be replaced by a new item, or some kind of restorative action (maintenance) is necessary to restore the efficiency of the whole system. The cost of maintenance depends upon a number of factors, and a stage comes at which the maintenance cost is so large that it is more profitable to replace the old item. Thus, there is a need to formulate the most effective replacement policy.

The purpose of this chapter is to show what replacement models look like.

Definition

Replacement models are concerned with the problem of replacement of machines, individuals, capital assets, etc. due to their deteriorating efficiency, failure, or breakdown.

It is evident that the study of replacement is a field of application rather than a method of analysis. Actually, it is concerned with methods of comparing **alternative** replacement policies.

The various types of replacement problems can be broadly classified in following categories:

- Replacement of items whose efficiency deteriorates with time, e.g., machine, tools, etc.
- Replacement of items that fail suddenly and completely like electric bulbs & tubes.
- Replacement of human beings in an organization or staffing problem.
- Replacement of items may be necessary due to new researches and methods; otherwise, the system may become outdated.

Replacement Of Items That Deteriorates With Time

We begin here with the simplest replacement model where the deterioration process is predictable. More complex replacement models are studied in the subsequent sections.

This model is represented by:

- Increasing maintenance cost.
- Decreasing salvage value.

Assumption

- Increased age reduces efficiency

Generally, the criteria for measuring efficiency is the discounted value of all future costs associated with each policy.

Let

C = the capital cost of a certain item, say a machine
S(t) = the selling or scrap value of the item after t years.
F(t) = operating cost of the item at time t
n = optimal replacement period of the time

Now, the annual cost of the machine at time t is given by C - S(t) + F(t) and since the total maintenance cost incurred on the machine during n years is $\int_0^n F(t)\,dt$, the total cost T, incurred on the machine during n years is given by:

$$T = C - S(t) + \int_0^n F(t)\,dt$$

Thus, the average annual total cost incurred on the machine per year during n years is given by

$$TA = \frac{1}{n}\left[C - S(t) + \int_0^n F(t)\,dt \right]$$

To determine the optimal period for replacing the machine, the above function is differentiated with respect to n and equated to zero.

$$\frac{dTA}{dn} = \frac{-1}{n^2}\left[C - S(t) \right] \frac{-1}{n^2} \int_0^n F(t)\,dt + \frac{F(n)}{n}$$

Equating $\frac{dTA}{dn} = 0$, we get

$$F(n) = \frac{1}{n}\left[C - S(t) + \int_0^n F(t)\,dt \right]$$

That is, F(n) = TA

Thus, we conclude that an item should be replaced when the average cost to date becomes equal to the current maintenance cost.

Examples:-

Constant Resale Value

Example

The initial cost of a machine is Rs. 7100 and scrap value is Rs. 100. The maintenance costs found from experience are as follows:

Year	1	2	3	4	5	6	7	8
Maintenance	200	350	500	700	1000	1300	1700	2100

When should the machine be **replaced**?

Solution:

Year	Running cost	Cumulative running cost	Scrap value	Difference between initial cost and scrap value	Average investment cost / year	Average running cost / year	Average annual total cost
A	**B**	**C**	**D**	**E**	**F = E/A**	**G = C/A**	**H = F + G**
1	200	200	100	7000	7000	200	7200
2	350	200 + 350 = 550	100	7000	3500	225	3775
3	500	550 + 500 = 1050	100	7000	2333.33	350	2683.33
4	700	1050 + 700 = 1750	100	7000	1750	437.5	2187.50
5	1000	1750 + 1000 = 2750	100	7000	1400	550	1950
6	1300	2750 + 1300 = 4050	100	7000	1166.67	675	1841.67
7	1700	4050 + 1700 = 5750	100	7000	1000	821.42	1821.42

8	2100	5750 + 2100 = 7850	100	7000	875	981.25	1856.25

This table shows that the average annual total cost during the seventh year is minimum. Hence, the machine should be **replaced** after the **7th** year.

Falling Resale Value

Example

The initial cost of a machine is Rs. 6100 and resale value drops as the time passes. Cost data are given in the following table:

Year	1	2	3	4	5	6	7	8
Maintenance	100	250	400	600	900	1200	1600	2000
Resale Value	800	700	600	500	400	300	200	100

When should the machine be replaced?

Solution:

Year	Running cost	Cumulative running cost	Resale value	Difference between initial cost and resale value	Average investment cost / year	Average running cost / year	Average annual total cost
1	100	100	800	5300	5300	100	5400
2	250	350	700	5400	2700	175	2875
3	400	750	600	5500	1833.33	250	2083.33
4	600	1350	500	5600	1400	337.5	1737.50
5	900	2250	400	5700	1140	450	1590
6	1200	3450	300	5800	966.67	575	1541.67
7	1600	5050	200	5900	842.85	721.42	1564.27
8	2000	7050	100	6000	750	881.25	1631.25

This table shows that the average annual total cost during the sixth year is minimum. Hence, the machine should be replaced after the 6th year.

Present Worth Factor

In this method, the present value of all future expenditures and revenues for each alternative is calculated. An item whose present worth factor is least is preferred.
Let

P = purchase cost of an item
A = annual running cost
n = life of an item in years
S = salvage value
r = annual interest rate

The present value can be calculated as follows:

P + A (Pwf for r% interest rate for n years) - S (Pwf for r% interest rate for n years)

For an illustration, consider the following problem.

Example

The China Moon restaurant is considering to purchase a new cooling system. Cost data are given in the following table:

	Cooling system A	Cooling system B	Cooling system C
Present investment (Rs.)	12000	14000	17000
Total annual cost (Rs.)	3000	2000	1500
Life (Years)	10	10	10
Salvage value (Rs.)	500	1000	1200

On the basis of above data, select the best cooling system considering 12% normal rate of return per year.
Given
Pwf (total annual cost) @ 12% for 10 years = 5.650
Pwf (salvage value) @ 12% for 10 years = 0.322

Solution:

	Cooling system A	Cooling system B	Cooling system C
Present investment (Rs.)	12000	14000	17000
Total annual cost (Rs.)	3000 X 5.650	2000 X 5.650	1500 X 5.650
Salvage value (Rs.)	500 X 0.322	1000 X 0.322	1200 X 0.322
Total Cost	28789	24978	25088.6

Total Cost = Present investment + Total annual cost - Salvage value
Cooling system A = 12000 + 16950 - 161 = Rs. 28789

Cooling system B = 14000 + 11300 - 322 = Rs. 24978
Cooling system C = 17000 + 8475 - 386.4 = Rs. 25088.6

Hence, cooling system B should be purchased because it has least total cost.

Replacement Of Items That Fail Completely

In some situations, failure of a certain item occurs all of a sudden, instead of gradual deterioration (e.g., failure of light bulbs, tubes, etc.). The failure of the item may result in complete breakdown of the system. The breakdown implies loss of production, idle inventory, idle labour, etc. Therefore, an organization must prepare itself against these **failures.**

Thus, to avoid the possibility of a complete breakdown, it is desirable to formulate a suitable replacement policy. The following two courses can be followed in such situations.

- **Individual replacement policy.** Under this policy, an item may be replaced immediately after its failure.
- **Group replacement policy**. Under this policy, the items are replaced in group after a certain period, say t, irrespective of the fact that items have failed or not. If any item fails before its preventive replacement is due, then individual replacement policy is used.

In situations where the items fail completely, the formulation of replacement policy depends upon the probability of failure. Mortality tables or Life testing techniques may be used to obtain a probability distribution of the failure of items in a system.

Mortality Tables

M(t) = Number of items surviving at time t
M(t - 1) = Number of items surviving at time (t - 1)
N = Total number of items in the system

The probability of failure of items during the interval t and (t - 1) is given by

$$\frac{M(t-1) - M(t)}{N}$$

The conditional probability that any item survived upto age (t - 1) and will fail in the next period is given by

M(t - 1) - M(t)

M(t - 1)

Example 1

Following mortality rates have been observed for certain type of light bulbs.

Time (weeks)	0	1	2	3	4	5	6	7	8	9	10
Number of bulbs still operating	100	94	82	58	40	28	19	13	7	3	0

Calculate the probability of failure.

Solution:

Here, t is the time (weeks) and M (t) is the number of bulbs still operating. The probability of failure can be calculated as shown in the following table.

Table

Time (t)	**M (t)**	**Probability of failure** p_i **= [M (t - 1)- M (t)] / N**
0	100	----
1	94	(100 - 94)/100 = 0.06
2	82	(94 - 82)/100 = 0.12
3	58	(82 - 58)/100 = 0.24
4	40	(58 - 40)/100 = 0.18
5	28	(40 - 28)/100 = 0.12
6	19	(28 - 19)/100 = 0.09
7	13	(19 - 13)/100 = 0.06
8	7	(13 - 7)/100 = 0.06
9	3	(7 - 3)/100 = 0.04
10	0	(3 - 0)/100 = 0.03

Example 2

Following mortality rates have been observed for a certain type of electronic component.

Month	0	1	2	3	4	5	6
% surviving at the end of the month	100	97	90	70	30	15	0

There are 10000 items in operation. It costs Re 1 to replace an individual item and 35 paise per item, if all items are replaced simultaneously. It is decided to replace all items at fixed intervals & to continue replacing individual items as and when they fail. At what intervals should all items be replaced? Assume that the items failing during a month are replaced at the end of the month.

Solution:

Month	% surviving at the end of the month	Probability of failure p_i
0	100	----
1	97	(100 - 97)/100 = 0.03
2	90	(97 - 90)/100 = 0.07
3	70	(90 - 70)/100 = 0.20
4	30	(70 - 30)/100 = 0.40
5	15	(30 - 15)/100 = 0.15
6	0	(15 - 0)/100 = 0.15

The given problem can be divided into two parts.

I. Individual replacement.
II. Group replacement.

Case I

It should be noted that no item survives for more than 6 months. Thus, an item which has survived for 5 months is sure to fail during sixth month.

The expected life of each item is given by

= $\sum x_i p_i$, where x_i is the month and p_i is the corresponding probability of failure.
= (1 X 0.03) + (2 X 0.07) + (3 X 0.20) + (4 X 0.40) + (5 X 0.15) + (6 X 0.15)
= 4.02 months.

∴ Average number of replacement every month = N/(average expected life) = 10000/4.02 = 2487.5
= 2488 items (approx.).
Here average cost of monthly individual replacement policy = 2488 X 1 = Rs. 2488/-, (the cost being Re 1/- per item).

Case II

Let N_i denote the number of items replaced at the end of ith month.

Calculating values for N_i

N_0 = Number of items in the beginning = 10,000
N_1 = Number of items during the 1st month X probability that an item fails during 1st month of installation
= 10000 X 0.03 = 300

N_2 = Number of items replaced by the end of second month
=(Number of items in beginning X probability that these items will fail in 2nd month) + (Number of items replaced in first month X probability that these items will fail during second month)

$=N_0P_2 + N_1P_1$
=(10000 X 0.07) + (300 X 0.03) = 709

$N_3 = N_0P_3 + N_1P_2 + N_2P_1$
= (10000 X 0.20) + (300 X 0.07)+ (709 X 0.03) = 2042

$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$
= (10000 X 0.40) + (300 X 0.20)+ (709 X 0.07) + (2042 X 0.03) = 4171

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$
= (10000 X 0.15) + (300 X 0.40)+ (709 X 0.20) + (2042 X 0.07) + (4171 X 0.03) = 2030

$N_6 = N_0P_6 + N_1P_5 + N_2P_4 + N_3P_3 + N_4P_2 + N_5P_1$
= (10000 X 0.15) + (300 X 0.15)+ (709 X 0.40) + (2042 X 0.20) + (4171 X 0.07) + (2030 X 0.03) = 2590.

From the above calculations, it is observed that N_i increases upto fourth month and then decreases. It can also be seen that N_i will later tend to increase and the value of N_i will oscillate till the system acquires a steady state.

The optimum replacement cycle under group replacement is given in the following table.

End of month	Total no. of items failed	Cumulative no. of failure	Cost of replacement after failure (Re 1/ item)	Cost of all replacement (Rs. 0.35/ item)	Total cost (Rs.)	Average cost per month (Rs.)
1	300	300	300	3500	3800	3800
2	709	1009	1009	3500	4509	2254.50
3	2042	3051	3051	3500	6551	2183.66
4	4171	7222	7222	3500	10722	2680.50
5	2030	9252	9252	3500	12752	2550.40
6	2590	11842	11842	3500	15342	2557.00

The above table shows that the average cost during the third month is Minimum. Thus, it would be economical to replace all the items every three months.

Staffing Problem

In the previous sections, we discussed about replacement problems, which were not related to human resources working in an organization. The replacement models can also be used to solve the problems of staff replacement. This section focuses on the problem of replacing staff in an organization. Staff replacement is essential due to the following factors:

- Inefficiency
- Resignation
- Retirement
- Unexpected events (like accident, death, etc.)

Assumption

- The life distribution for the service of staff in an organization is predetermined.

Example 1

A team of software developers at www.universalteacher.com is planned to rise to a strength of 50 persons, and then to remain at that level. Consider the following data:

Year	Total % who have left upto the end of the year
1	5
2	30
3	50
4	60
5	70
6	75
7	80
8	85
9	90
10	100

On the basis of above information, determine:

What is the recruitment per year necessary to maintain the strength? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant can expect his promotion to one of these posts?

Solution:

Calculating values for table 1

- The values in column (c) are obtained by subtracting the corresponding elements of column (b) from 100.
- The values in column (d) are obtained by dividing the corresponding elements in column (b) by 100.
- The values in column (e) are obtained by dividing the corresponding elements in column (c) by 100.

Table 1

Year	No.of persons who leave at the end of the year	No. of persons in service at the end of year	Prob. of leaving at the end of the year	Prob. of in service at the end of the year
a	**b**	**c**	**d**	**e**
0	0	100	0	1.00
1	5	95	0.05	0.95
2	30	70	0.30	0.70
3	50	50	0.50	0.50
4	60	40	0.60	0.40
5	70	30	0.70	0.30
6	75	25	0.75	0.25
7	80	20	0.80	0.20
8	85	15	0.85	0.15
9	90	10	0.90	0.10
10	100	0	1.00	0
Total		**455**		

From table 1, we find that with a recruitment policy of 100 persons every year, the total number of persons serving in the organization would have been 455. Hence, if we want to maintain a strength of 50 persons then we should recruit

$$\frac{100 \times 50}{455} = \frac{1000}{91} = 10.98 = 11 \text{ persons/year}$$

Every year 11 persons should be **recruited** to maintain a strength of 50. Number of survivals after each year can be obtained by multiplying the various values of column (e) by 11.

Table 2

Year	Number of persons in service
0	11
1	10
2	8
3	6
4	4
5	3
6	3

7	2
8	2
9	1
10	0

Now there are 8 senior posts. From table 2, it can seen that there are 3 persons in service during the sixth year, 2 in seventh year, 2 in eighth year, and 1 in ninth year. Hence, promotions of new recruits will start by the end of sixth year and will continue upto seventh year.

Example 2

The Railway Ministry requires 200 private assistants, 300 private secretaries, and 50 section officers. Persons are recruited at the age of 21, if still in service, retire at the age of 60. Given the following life table, determine

- How many persons should be recruited every year ?
- At what age promotions should take place ?

Age	21	22	23	24	25	26	27	28
No. in service	1000	600	400	380	311	260	229	206
Age	29	30	31	32	33	34	35	36
No. in service	190	180	174	166	162	155	150	146
Age	37	38	39	40	41	42	43	44
No. in service	145	135	131	125	120	112	105	100
Age	45	46	47	48	49	50	51	52
No. in service	94	86	80	73	65	60	53	46
Age	53	54	55	56	57	58	59	60
No. in service	40	32	26	23	19	13	11	0

Solution:

If a policy of recruiting 1000 persons every year is followed, then the total number of employees in service between the age 21 to 59 years will be equal to 6403. But the requirement of organization is 550 (200 + 300 + 50) employees.

Therefore, to maintain a strength of 550 employees, the organization should recruit: (1000 X 550)/6403 = 86 (approx.) persons every year.

Private assistants

Out of a strength of 550, there are 200 private assistants. Hence, out of a strength of 1000 there will be
(200 X 1000)/550 = 364 private assistants.
From the above life table, 364 is available upto 24 years. Therefore, the promotion of private assistants will take place in 25th year.

Private Secretaries

Out of a strength of 1000 there will be
(300 X 1000)/550 = 545 private secretaries.

Section officers

Number of section officers = 1000 - (364 + 545) = 91.
From the above life table, we find that at the age of 46 only 86 will survive. Therefore, promotion of private secretaries will take place in 46th year.

Self Test Questions

Theory

1. What is replacement? Describe some important replacement situations.

2. Explain the different types of replacement models.

3. Write a short note on staffing problem.

Practical

1. A firm is considering replacement of a machine whose cost is Rs. 12,200 and the scrap value is Rs. 200. The maintenance cost found from experience to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced?

2. The initial cost of a machine is Rs. 6100 and resale value drops as the time passes. Cost data are given in the following table:

Year	1	2	3	4	5	6	7	8
Maintenance	100	250	400	600	900	1200	1600	2000
Resale Value	800	700	600	550	500	450	410	380

When should the machine be replaced?

3. The following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5000.

Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Maintenance Cost (Rs.)	1500	1600	1800	2100	2500	2900	3400	4000
Resale Value (Rs.)	3500	2500	1700	1200	800	500	500	500

At what year is the replacement due?

4. The following mortality rates have been observed for an electronic component:

Month	**1**	**2**	**3**	**4**	**5**	**6**
% failure at the end of the month	8	22	45	70	85	100

There are 1500 items in operation. It costs Rs. 20/- to replace an individual item and 50 paise per item, if all items are replaced simultaneously. It is decided to replace all items at fixed intervals and to continue replacing individual items as and when they fall. At what intervals should all items be replaced?

5. The following mortality rates have been observed for a certain type of light bulbs:

Week	**1**	**2**	**3**	**4**	**5**
% failure at the end of the week	10	25	50	80	100

There are 1000 bulbs in use, and it costs Rs. 1/- to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it would cost 25 paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced?

6. A team of software developers at www.universalteacher.com is planned to rise to a strength of 50 persons, and then to remain at that level. Consider the following data:

Year	Total % who have left upto the end of the year
1	5
2	36
3	56
4	63
5	70
6	75
7	80
8	85
9	90
10	100

On the basis of above information, determine:

What is the recruitment per year necessary to maintain the strength? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant can expect his promotion to one of these posts?

Chapter – 14

Sequencing Models

Introduction

In the previous chapter, you studied the mechanics of obtaining an optimal replacement policy for machines. This chapter concentrates on the problem of determining the sequence (order) in which a number of jobs should be performed on different machines in order to make effective use of available facilities and achieve greater output.

For example, consider a sequencing problem where n jobs are to be performed on m different machines. In such a case, our problem is to determine the sequence, which **minimizes the total elapsed time**. Here, the term elapsed time means the time from the start of first job upto the completion of the last job.

In this chapter, we discuss the following cases:

- Processing n jobs through two machines.
- Processing n jobs through three machines.
- Processing two jobs through m machines.

What are the underlying assumptions?

Few general assumptions in this chapter are as follows:

- The processing time on each machine is known.
- The time required to complete a job is independent of the order of the jobs in which they are to be processed.
- No machine may process more than one job simultaneously.
- The time taken by each job in changing over from one machine to another is negligible.
- Each job, once started on a machine is to be performed up to completion on that machine.
- The order of completion of job has no significance, i.e., no job is to be given priority.
- A job starts on the machine as soon as the job and the machine both are idle.

Taxonomy Of Sequencing Models

Before examining the solution of specific sequencing models, you will find it useful to have an overview of such systems. This section classifies the sequencing problems.

Job Arrival Pattern

The usual pattern of arrivals into the system may be static or dynamic.

Static: If certain numbers of jobs arrive simultaneously and no further jobs arrive until the present set of jobs has been processed, then the problem is said to be static.
Dynamic: In this case, jobs arrive after certain interval of time and arrival of jobs will continue indefinitely in future also.

Number of Machines

A sequencing problem may be called single processor or multiple processor problem, according to the number of machines available in the shop. The multiple processor case may be further classified as

- parallel
- series
- hybrid

Sequence of Machines

Fixed sequence: In this case, given jobs are processed in a fixed order. An example of such a case will be where each job is to be processed first on machine 1, then on machine 2 then on machine 3, and so on.
Random Sequence: In this case, given jobs are processed in a random order.

Processing Time

Deterministic: If the processing time is known with certainty, it is a deterministic problem.
Probabilistic: If only expected processing time is known, then it is a probabilistic problem.

Processing n Jobs Through Two Machines

This problem refers to the following situation:
Suppose n jobs are to be processed on two machines, say A & B. Each job has to pass

through the same sequence of operations in the same order, i.e., passing is not allowed. After a job is completely processed on machine A, it is assigned to machine B. If machine B is not free at that moment, then the job enters the waiting queue. Each job from the waiting queue is assigned to machine B according to FIFO discipline. Let
A_i = Processing time for ith job on machine A
B_i = Processing time for ith job on machine B
T = Total elapsed time

The problem here is to determine the sequence in which these n jobs should be processed through A & B, so that the total elapsed time (T) is Minimum.

Optimal Sequence Algorithm

The best technique for determining an optimal sequence was developed by Johnson & Bellman, which is discussed below.

Steps

1. Select the minimum processing time out of all the A_i's and B_i's. If it is A_r then do the rth job first. If it is B_s then do the sth job in last.
2. If there is a tie in selecting minimum of all the processing times, then there are following three ways to deal with such a situation:
    - If the minimum of all the processing times is A_r, which is also equal to B_s. That is, Min $(A_i, B_i) = A_r = B_s$
      Then do the rth job first and sth job in last.
    - If Min $(A_i, B_i) = A_r$, but $A_r = A_k$, i.e., there is a tie for minimum among A_i's, then select any one.
    - If Min $(A_i, B_i) = B_s$, but $B_s = B_t$, i.e., there is a tie for minimum among B_i's, then select any one.
3. Now eliminate the job which has already been assigned from further consideration, and repeat steps 1 and 2 until an optimal sequence is found.

The next section concentrates on how to calculate the total elapsed time.

Example 1

Suppose we have five jobs, each of which has to be processed on two machines A & B in the order AB. Processing times are given in the following table:

Job	Machine A	Machine B
1	6	3
2	2	7
3	10	8
4	4	9
5	11	5

Determine an order in which these jobs should be processed so as to minimize the total processing time.

Solution:

The minimum time in the above table is 2, which corresponds to job 2 on machine A.

2				

Now we eliminate job 2 from further consideration. The reduced set of processing times are as follows:

Job	Machine A	Machine B
1	6	3
3	10	8
4	4	9
5	11	5

The minimum time is 3 for job 1 on machine B. Therefore, this job would be done in last. The allocation of jobs till this stage would be

2				1

After deletion of job 1, the reduced set of processing times are as follows:

Job	Machine A	Machine B
3	10	8
4	4	9
5	11	5

Similarly, by repeating the above steps, the optimal sequence is as follows:

2	4	3	5	1

Once the optimal sequence is obtained, the minimum elapsed time may be calculated as follows:

Job	Machine A		Machine B	
	Time in	Time out	Time in	Time out
2	0	2	2	9
4	2	6	9	18
3	6	16	18	26
5	16	27	27	32
1	27	33	33	36

Idle time for machine A = total elapsed time - time when the last job is out of machine A
36 - 33 = 3 hours
Idle time for machine B = Time at which the first job in a sequence finishes on machine A + $\sum_{i=2}^{n}$ {(time when the ith job starts on machine B) - (time when the (i -1)th finishes on machine B)}

Idle time for machine B = 2 + (9 - 9) + (18 - 18) + (27 - 26) + (33 - 32) = 4 hours

Example 2

Strong Book Binder has one printing machine, one binding machine, and the manuscripts of a number of different books. Processing times are given in the following table:

Book	Time In Hours	
	Printing	Binding
A	5	2
B	1	6
C	9	7
D	3	8
E	10	4

We wish to determine the order in which books should be processed on the machines, in order to minimize the total time required.

Solution:

The minimum time in the above table is 1, which corresponds to the book B on printing machine.

B				

Now book B is eliminated. The reduced set of processing times are as follows:

Book	Time In Hours	
	Printing	Binding
A	5	2
C	9	7
D	3	8
E	10	4

The minimum time is 2 for book A on binding machine. Therefore, this job should be done in last. The allocation of jobs till this stage is:

B				A

The reduced set of processing times are as follows:

Book	Time In Hours	
	Printing	Binding
C	9	7
D	3	8
E	10	4

Similarly, by repeating the above steps, the optimal sequence is as follows:

B	D	C	E	A

Once the optimal sequence is obtained, the minimum elapsed time may be calculated as follows:

Book	Printing		Binding	
	Time in	Time out	Time in	Time out
B	0	1	1	7
D	1	4	7	15
C	4	13	15	22
E	13	23	23	27
A	23	28	28	30

Idle time for printing process = total elapsed time - time when the last job is out of machine A
30 - 28 = 2 hours
Idle time for binding process = 1 + (7 - 7) + (15 - 15) + (23 - 22) + (28 - 27) = 3 hours

Processing n Jobs Through Three Machines

This case is similar to the previous case except that instead of two machines, there are three machines. Problems falling under this category can be solved by the method developed by Johnson. Following are the two conditions of this approach:

- The smallest processing time on machine A is greater than or equal to the greatest processing time on machine B, i.e.,

$$\text{Min. } (A_i) \geq \text{Max. } (B_i)$$

- The smallest processing time on machine C is greater than or equal to the greatest processing time on machine B, i.e.,

$$\text{Max. } (B_i) \leq \text{Min. } (C_i)$$

If **either or both** of the above conditions are satisfied, then we replace the three machines by two fictitious machines G & H with corresponding processing times given by

$G_i = A_i + B_i$
$H_i = B_i + C_i$

Where G_i & H_i are the processing times for ith job on machine G and H respectively. After calculating the new processing times, we determine the optimal sequence of jobs for the machines G & H in the usual manner.

Example 1

The MDH Masala company has to process five items on three machines:- A, B & C. Processing times are given in the following table:

Item	A_i	B_i	C_i
1	4	4	6
2	9	5	9
3	8	3	11
4	6	2	8
5	3	6	7

Find the sequence that minimizes the total elapsed time.

Solution:

Here, Min. $(A_i) = 3$, Max. $(B_i) = 6$ and Min. $(C_i) = 6$. Since the condition of Max. $(B_i) \leq$ Min. (C_i) is satisfied, the problem can be solved by the above procedure. The processing times for the new problem are given below.

Item	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	8	10
2	14	14
3	11	14
4	8	10
5	9	13

The optimal sequence is

1	4	5	3	2

Item	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
1	0	4	4	8	8	14
4	4	10	10	12	14	22
5	10	13	13	19	22	29
3	13	21	21	24	29	40
2	21	30	30	35	40	49

Total elapsed time = 49

Idle time for machine A = 49 – 30 = 19 hours
Idle time for machine B = 4 + (10 - 8) + (13 - 12) + (21 - 19)+ (30 - 24) + (49 - 35) = 29 hours
Idle time for machine C = 8 + (14 - 14) + (22 - 22) + (29 - 29)+ (40 - 40) = 8 hours

Example 2

Shahi Export House has to process five items through three stages of production, viz, cutting, sewing & pressing. Processing times are given in the following table:

Item	Cutting (A_i)	Sewing (Bi)	Pressing (C_i)
1	3	3	5
2	8	4	8
3	7	2	10
4	5	1	7
5	2	5	6

Determing an order in which these items should be processed so as to minimize the total processing time.

Solution:

The processing times for the new problem are given below.

Item	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	6	8
2	12	12
3	9	12
4	6	8
5	7	11

Thus, the optimal sequence may be formed in any of the two ways.

1	4	5	3	2

4	1	5	3	2

Item	Cutting		Sewing		Pressing	
	Time in	Time out	Time in	Time out	Time in	Time out
1	0	3	3	6	6	11
4	3	8	8	9	11	18
5	8	10	10	15	18	24
3	10	17	17	19	24	34
2	17	25	25	29	34	42

Total elapsed time = 42

Idle time for cutting process = 42 – 25 = 17 hours
Idle time for sewing process = 3 + (8 - 6) + (10 - 9) + (17 - 15)+ (25 - 19) + (42 - 29) = 27 hours.
Idle time for pressing process = 6 + (11 - 11) + (18 - 18) + (24 - 24)+ (34 - 34) = 6 hours.

Processing 2 Jobs through m Machines

This section focuses on the problem of processing two jobs through m machines. Problems under this category can be solved with the help of graphical method. The graphical method is explained with the help of the following example.

Example

Two jobs are to be performed on five machines A, B, C, D, and E. Processing times are given in the following table.

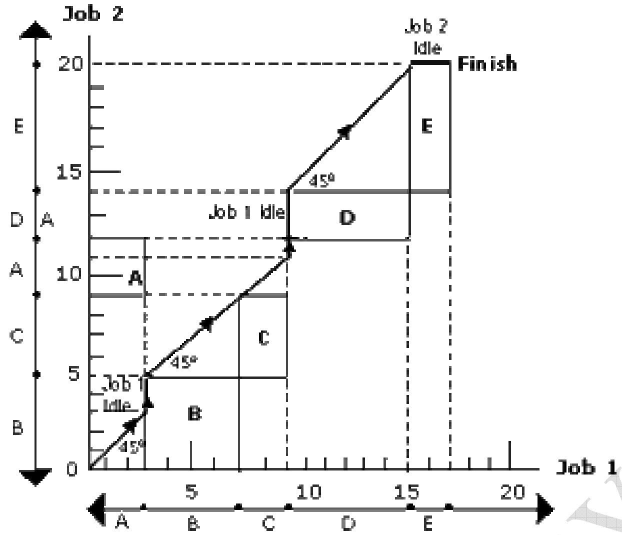
			Machine				
Job 1	Sequence	:	A	B	C	D	E
	Time	:	3	4	2	6	2
Job 2	Sequence	:	B	C	A	D	E
	Time	:	5	4	3	2	6

Use graphical method to obtain the total minimum elapsed time.

Solution:

Steps

- Mark the processing times of job 1 & job 2 on X-axis & Y-axis respectively.
- Draw the rectangular blocks by pairing the same machines as shown in the following figure.



- Starting from origin O, move through the 45^0 line until a point marked finish is obtained.
- The elapsed time can be calculated by adding the idle time for either job to the processing time for that job. In this illustration, idle time for job 1 is 5 (3+2) hours.
  Elapsed time = Processing time of job 1 + Idle time of job 1
  = (3 + 4 + 2 + 6 + 2) + 5 = 17 + 5 = 22 hours.
  Likewise, idle time for job 2 is 2 hours.
  Elapsed time = Processing time of job 2 + Idle time of job 2
  = (5 + 4 + 3 + 2 + 6) + (2) = 20 + 2 = 22 hours.

Self Test Questions

Theory

1. What do you understand by sequencing problem?

2. Write short notes on the following:

- Sequencing decision problem for n jobs on two machines.
- Sequencing decision problem for n jobs on three machines.

Practical

1. Shahi Export House has to process five items through two stages of production, viz, cutting & sewing. Processing times are given in the following table:

Items	Time In Hours	
	Cutting	Sewing
A	7	4
B	3	8
C	11	9
D	5	10
E	12	6

Determine an order in which these items should be processed so as to minimize the total processing time.

2. Find the sequence that minimizes the total elapsed time required to complete the following jobs.

Processing time	Job	1	2	3	4	5	6
	Ai	3	6	5	4	3	2
	Bi	7	9	2	3	4	6

3. A book binder has one printing press, one binding machine, and the manuscripts of a number of different books. The times required to perform the printing and binding operations for each book are known. We wish to determine the order in which books should be processed on the machines, in order to minimize the total time required.

Books	Printing time	Binding time
1	30	80
2	120	100
3	50	90
4	20	60
5	90	30
6	110	10

4. Six jobs go first over machine I and then over II. The order of the completion of jobs has no significance. The following table gives the machine time in hours for six jobs and the two machines:

Job no.	1	2	3	4	5	6
Machine I	7	11	6	9	10	8
Machine II	9	6	10	5	11	7

Find the sequence of the jobs that minimizes the total elapsed time to complete the jobs.

5. Find the sequence that minimizes the total elapsed time required to complete the following jobs:

	Processing times in hours					
No. of job	**1**	**2**	**3**	**4**	**5**	**6**
Machine A	4	8	3	6	7	5
Machine B	6	3	7	2	8	4

6. We have five jobs, each of which must go through the two machines in the order AB. Processing times are given below:

No. of job	1	2	3	4	5
Machine A	9	1	17	5	19
Machine B	3	11	13	15	7

Determine a sequence for the five jobs that will minimize the total elapsed time.

7. Find the sequence that minimizes the total elapsed time (in hours) required to complete all the following jobs on machines M_1, M_2 and M_3 in the order $M_1M_2M_3$.

No. of job	A	B	C	D	E
M_1	4	9	8	6	5
M_2	5	6	2	3	4
M_3	8	10	6	7	11

8. Find the sequence that minimizes the total time required for performing the following jobs on three machines in the order ABC.

Processing times			
Jobs	**Machine A**	**Machine B**	**Machine C**
1	8	3	8
2	3	4	7
3	7	5	6
4	2	2	9
5	5	1	10
6	1	6	9

9. We have five jobs, each of which must go through the machines A, B and C in the order ABC:

Processing times in hours					
Job no.	(1)	(2)	(3)	(4)	(5)
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

Determine a sequence for the jobs that will minimize the total elapsed time.

10. Two jobs are to be performed on five machines A, B, C, D, and E. Processing times are given in the following table.

			Machine				
Job 1	**Sequence**	**:**	A	B	C	D	E
	Time	**:**	4	5	3	7	3
Job 2	**Sequence**	**:**	B	C	A	D	E
	Time	**:**	6	5	4	3	7

Use graphical method to obtain the total minimum elapsed time.

Chapter – 15

Nonlinear Programming

Introduction

A linear programming problem is characterized by the presence of linear constraints and linear objective function in decision variables. A linear programming problem can be viewed as

Optimize (maximize or minimize) $\sum_{j=1}^{n} c_j x_j$

Subject to

$\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq)\ b_i;\ i = 1, 2, \ldots, m$

$x_j \geq 0;\ j = 1, 2, \ldots, n$

There are, however, problems in real life situations where neither the objective function nor the constraints are linear functions in decision variables. For example, in a model for a steel-processing plant, a variable representing the temperature of a blast furnance can be described by a nonlinear function of variables indicating the amount and duration of heat energy applied. Each of these variables, in turn, is contained in other constraints as well as in the objective function. The term nonlinear programming usually refers to problems such as

Maximize $c\,(x_1, x_2, \ldots, x_n)$

Subject to
$a_i\,(x_1, x_2, \ldots, x_n) \leq 0$, for $i = 1, 2, \ldots, m$

Where both $c\,(x_1, x_2, \ldots, x_n)$ and $a_i\,(x_1, x_2, \ldots, x_n)$ are real-valued, nonlinear functions of n real variables.

Introduction To Quadratic Programming

Definition

A **quadratic programming problem** is a mathematical programming problem, which consists of an objective function composed of linear and quadratic terms and a set of linear constraints.

A quadratic programming problem can be solved by a modification of the simplex procedure.

The general quadratic programming problem is given by

$$\text{Maximize } f = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \le 0 \text{ , } i = 1, 2, ..., m$$

$$x_j \ge 0, \ j = 1, 2, ..., n$$

If the quadratic form is **negative semidefinite and symmetric**, then it is a **concave function** of the decision variables. Since the constraints are **linear**, the feasible region is **convex**. Thus, Kuhn-Tucker conditions in this case are both necessary and sufficient.

To derive these conditions, we use multipliers λ_i (i = 1, 2, ..., m) corresponding to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \le 0;$$

and μ_j (j = 1, 2, ..., n) corresponding to the non-negativity constraints.

$$df/dx_j = c_j + 2 \sum_{k=1}^{n} c_{jk} x_k \text{ , } j = 1, 2, ..., n$$

$$\frac{d}{dx_j} \left[\sum_{j=1}^{n} a_{ij} x_j - b_i \right] = a_{ij} \text{ ; } i = 1, 2, ..., m; \ j = 1, 2, ..., n$$

Let $S_i = b_i - \sum_{j=1}^{n} a_{ij}x_j$; i = 1, 2, ..., m .....(i)
Where S_i is a slack variable.

Hence, Kuhn-Tucker conditions are given by

$$c_j + 2\sum_{k=1}^{n} c_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0;\ j = 1, 2, ..., n$$

$$\text{or } -2\sum_{k=1}^{n} c_{jk}x_k + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j = c_j;\ j = 1, 2, ..., n \text{ .....(ii)}$$

$S_i\lambda_i = 0$, $x_j\mu_j = 0$
$S_i, \lambda_i, \mu_j, x_j \geq 0$, i = 1, 2, ..., m; j = 1, 2, ..., n

If we ensure that the pairs (λ_i, S_i) and (μ_j, x_j) are not into the basis simultaneously, then the conditions $S_i\lambda_i = 0$ and $x_j\mu_j = 0$ will be automatically satisfied. Therefore, we use simplex procedure with a restricted entry rule.

Quadratic Simplex Method

Example

Maximize $f(x) = 2x_1 + 3x_2 - x_1^2 - x_2^2$

subject to
$x_1 + x_2 \leq 2$
$2x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$.

Solution.

First, find that a given function is concave or convex.
$df(x)/dx_1 = 2 - 2x_1$
$df(x)/dx_2 = 3 - 2x_2$
$d^2f(x)/dx_1^2 = -2$

$d^2f(x)/dx_2^2 = -2$
$d^2f(x)/dx_1.dx_2 = 0$

H(x) =	$d^2f(x)/dx_1^2$	$d^2f(x)/dx_1.dx_2$
	$d^2f(x)/dx_1.dx_2$	$d^2f(x)/dx_2^2$

H(x) =	– 2	0
	0	–2

I =	1	0
	0	1

λ(I)	λ	0
	0	λ

H(x) – λ(I)	– 2	0	–	λ	0
	0	–2		0	λ

H(x) – λ(I)	– 2 – λ	0
	0	– 2 – λ

Determinant of H(x) – λ(I) = 0

$(-2-\lambda) \times (-2-\lambda) - 0 \times 0 = 0$
$\lambda^2 + 4\lambda + 4 = 0$
$\lambda^2 + 2\lambda + 2\lambda + 4 = 0$
$\lambda(\lambda + 2) + 2(\lambda + 2) = 0$
$\lambda = -2$

Since λ is negative, the given problem has a concave objective function.

$\phi(x, \lambda) = 2x_1 + 3x_2 - x_1^2 - x_2^2 + \lambda_1(2 - x_1 - x_2) + \lambda_2(3 - 2x_1 - x_2)$

Differentiate w.r.t. x_1
$\phi x_1 = 2 - 2x_1 - \lambda_1 - 2\lambda_2 = -\mu_1$.....(i)

Differentiate w.r.t. x_2
$\phi x_2 = 3 - 2x_2 - \lambda 1 - \lambda 2 = -\mu_2$.....(ii)

Where μ_1 and μ_2 are surplus variables.

$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$
$\mu_1 x_1 = 0, \mu_2 x_2 = 0$

Differentiate w.r.t. λ_1
$\phi \lambda_1 = 2 - x_1 - x_2 = S_1$.....(iii)

Differentiate w.r.t. λ_2
$\phi \lambda_2 = 3 - 2x_1 - x_2 = S_2$.....(iv)

Where S_1 and S_2 are slack variables.

$\lambda_1 S_1 = 0, \lambda_2 S_2 = 0$

Kuhn Tucker Conditions are given by

From equations (i), (ii), (iii) & (iv), we get

$2x_1 + \lambda_1 + 2\lambda_2 - \mu_1 = 2$
$2x_2 + \lambda_1 + \lambda_2 - \mu_2 = 3$
$x_1 + x_2 + S_1 = 2$
$2x_1 + x_2 + S_2 = 3$

Introducing artificial variables A_1 and A_2 to the first two equations to obtain a feasible solution of the problem.

Maximize $-A_1 - A_2$

Subject to

$2x_1 + \lambda_1 + 2\lambda_2 - \mu_1 + A_1 = 2$
$2x_2 + \lambda_1 + \lambda_2 - \mu_2 + A_2 = 3$
$x_1 + x_2 + S_1 = 2$
$2x_1 + x_2 + S_2 = 3$

Now, we solve the above problem by **Simplex method**.

Table 1

	c_j	0	0	0	0	0	0	0	0	–1	–1	
c_B	Basic variables B	x_1	x_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	A_1	A_2	Solution values b (=X_B)
–1	A_1	2	0	–1	0	0	0	1	2	1	0	2
–1	A_2	0	2	0	–1	0	0	1	1	0	1	3
0	S_1	1	1	0	0	1	0	0	0	0	0	2
0	S_2	2	1	0	0	0	1	0	0	0	0	3
z_j–c_j		–2	–2	1	1	0	0	–2	–3	0	0	

If we ensure that the pairs (λ_i, S_i) and (μ_j, x_j) are not basic variables simultaneously then the conditions $S_i\lambda_i = 0$ and $x_j\mu_j = 0$ will be automatically satisfied. Although $z_j – c_j$ is lowest corresponding to λ_2 column, it can't be made a basic variable as S_2 is already a basic variable. Hence, A_1 departs and x_1 enters.

Table 2

	c_j	0	0	0	0	0	0	0	0	–1	
c_B	Basic variables B	x_1	x_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	A_2	Solution values b (=X_B)
0	x_1	1	0	–1/2	0	0	0	1/2	1	0	1
–1	A_2	0	2	0	–1	0	0	1	1	1	3
0	S_1	0	1	1/2	0	1	0	–1/2	–1	0	1
0	S_2	0	1	1	0	0	1	–1	–2	0	1
z_j–c_j		0	–2	0	1	0	0	–1	–1	0	

Table 3

	c_j	0	0	0	0	0	0	0	0	–1	
c_B	Basic Variables B	x_1	x_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	A_2	Solution values b (=X_B)
0	x_1	1	0	–1/2	0	0	0	1/2	1	0	1
–1	A_2	0	0	–1	–1	–2	0	2	3	1	1
0	x_2	0	1	1/2	0	1	0	–1/2	–1	0	1
0	S_2	0	0	1/2	0	–1	1	–1/2	–1	0	0
z_j–c_j		0	0	1	1	2	0	–2	–3	0	

Since S_2 is a basic variable, λ_2 can't be made a basic variable.
Therefore, the variable A_2 departs and λ_1 enters.

Table 4

	c_j	0	0	0	0	0	0	0	0	
c_B	Basic variables B	x_1	x_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	Solution values b ($=X_B$)
0	x_1	1	0	–1/4	1/4	1/2	0	0	1/4	3/4
0	λ_1	0	0	–1/2	–1/2	–1	0	1	3/2	1/2
0	x_2	0	1	1/4	–1/4	1/2	0	0	–1/4	5/4
0	S_2	0	0	1/4	–1/4	–3/2	1	0	–1/4	1/4
$z_j–c_j$		0	0	0	0	0	0	0	0	

The values for x_1 and x_2 are 3/4 and 5/4 respectively.
The associated optimal value of the objective function is $f(x) = 2 \text{ X } 3/4 + 3 \text{ X } 5/4 – (3/4)^2 – (5/4)^2 = 25/8$

Separable Programming

Separable programming is an approximate method for solving nonlinear problems. It involves a minor modification of the simplex technique. This technique can be applied to problems in which all the nonlinear functions are separable. A separable function can be expressed as the sum of sub-functions where each sub-function is a function of one variable only.

The main idea behind this technique is to construct a constrained optimization model that linearly approximates the original nonlinear problem. This technique has a considerable practical significance because after breaking the problem, usual simplex method with certain modifications can be applied. But this technique increases the computational burden because converting a nonlinear problem into an approximate version with separable functions increases the size of the model.

The general form of a separable programming problem is given by

Maximize f_0 (x_1, x_2, ..., x_n)

Subject to

f_1 (x_1, x_2, ..., x_n) $\le b_i$; i = 1, 2, ..., m
$x_j \ge 0$, j = 1, 2, ..., n

If the objective function and the constraints are separable, then we can write

$$f_0 (x_1, x_2, \ldots\ldots, x_n) = \sum_{j=1}^{n} f_{0j} (x_j)$$

$$f_i (x_1, x_2, \ldots\ldots, x_n) = \sum_{j=1}^{n} f_{ij} (x_j)$$

$i = 1, 2, \ldots\ldots, m$

Thus, the problem under consideration can be expressed as

Maximize $\sum_{j=1}^{n} f_{0j} (x_j)$

Subject to

$$\sum_{j=1}^{n} f_{ij} (x_j) \leq b_i;\ i = 1, 2, \ldots\ldots, m$$

$x_j \geq 0$

Suppose there are two break points. We can evaluate $f_{ij} (x_j)$ at two consecutive break points and then make a linear approximation of $f_{ij} (x_j)$ between $x_j = a_j^k$ and $x_j = a_j^{k+1}$ in terms of two end points $f_{ij} (a_j^k)$ and $f_{ij} (a_j^{k+1})$ as:

$$f_{ij} (x_j) = W_j^k f_{ij} (a_j^k) + W_j^{k+1} f_{ij} (a_j^{k+1})$$

$$a_j^k \leq x_j \leq a_j^{k+1}$$

$$W_j^k + W_j^{k+1} = 1,\ W_j^k \geq 0$$

In general, there are k_j break points for the sub-function rather than two. The function $f_{ij} (x_j)$ can be expressed as

$$f_{ij} (x_j) = \sum_{k=1}^{k_j} W_j^k f_{ij} (a_j^k)$$

In the above expression, not more than **two W_j^k can be positive**. And if two are positive, then they must be **adjacent**.

$W_j^k \geq j = 1, 2, ..., n; k = 1, 2, ..., k_j$

$\sum_{k=1}^{kj} W_j^k = 1$

Thus, the technique of separable programming transforms the original nonlinear programming problem into a linear programming problem with decision variables W_j^k.

Separable Programming-Example

Example

Maximize $f_0 (x_1, x_2) = 5x_1 - x_1^2 + 3x_2 - x_2^2$

subject to

$f_1 (x_1, x_2) = 2x_1^4 + x_2 \leq 32$
$f_2 (x_1, x_2) = x_1 + 2x_2^2 \leq 32$

$x_1, x_2 \geq 0$
The break points of x_1 are 0, 1, 2 and x_2 are 0, 1, 2, 3, 4.

Solution:

$f_{01} (x_1) = 5x_1 - x_1^2$
$f_{11} (x_1) = 2x_1^4$
$f_{21} (x_1) = x_1$

$f_{02} (x_2) = 3x_2 - x_2^2$
$f_{12} (x_2) = x_2$
$f_{22} (x_2) = 2x_2^2$

a_1^k	$f_{01}(a_1^k)$	$f_{11}(a_1^k)$	$f_{21}(a_1^k)$	Weight
0	0	0	0	W_1^1
1	4	2	1	W_1^2
2	6	32	2	W_1^3

a_2^k	$f_{02}(a_2^k)$	$f_{12}(a_2^k)$	$f_{22}(a_2^k)$	Weight
0	0	0	0	W_2^1
1	2	1	2	W_2^2
2	2	2	8	W_2^3
3	0	3	18	W_2^4
4	-4	4	32	W_2^5

The linear model can be written as:

Maximize $0W_1^1 + 4W_1^2 + 6W_1^3 + 0W_2^1 + 2W_2^2 + 2W_2^3 + 0W_2^4 - 4W_2^5$

Subject to

$0W_1^1 + 2W_1^2 + 32W_1^3 + 0W_2^1 + W_2^2 + 2W_2^3 + 3W_2^4 + 4W_2^5 + S_1 = 32$
$0W_1^1 + W_1^2 + 2W_1^3 + 0W_2^1 + 2W_2^2 + 8W_2^3 + 18W_2^4 + 32W_2^5 + S_2 = 32$
$W_1^1 + W_1^2 + W_1^3 = 1$
$W_2^1 + W_2^2 + W_2^3 + W_2^4 + W_2^5 = 1$

$W_j^k \geq 0.$

Recall that for any j not two W_j^k can be positive; and if two are positive they must be adjacent. The successive simplex tables are given below:

Table 1

	c_j	0	4	6	0	2	2	0	-4	0	0	
c_B	**Basic Variables B**	W_1^1	W_1^2	W_1^3	W_2^1	W_2^2	W_2^3	W_2^4	W_2^5	S_1	S_2	**Solution Values b=(X_B)**
0	S_1	0	2	32	0	1	2	3	4	1	0	32
0	S_2	0	1	2	0	2	8	18	32	0	1	32
0	W_1^1	1	1	1	0	0	0	0	0	0	0	1
0	$W_2$1	0	0	0	1	1	1	1	1	0	0	1
z_j-c_j		0	-4	-6	0	-2	-2	0	4	0	0	

W_1^1 departs and W_1^3 enters.

Table 2

	c_j	0	4	6	0	2	2	0	-4	0	0	
c_B	Basic Variables B	W_1^1	W_1^2	W_1^3	W_2^1	W_2^2	W_2^3	W_2^4	W_2^5	S_1	S_2	Solution Values b=(X_B)
0	S_1	-32	-30	0	0	1	2	3	4	1	0	0
0	S_2	-2	-1	0	0	2	8	18	32	0	1	30
6	W_1^3	1	1	1	0	0	0	0	0	0	0	1
0	W_2^1	0	0	0	1	1	1	1	1	0	0	1
z_j-c_j		6	2	0	0	-2	-2	0	4	0	0	

S_1 is replaced by W_2^2. This is possible since W_2^1 is a basic variable.

Table 3

	c_j	0	4	6	0	2	2	0	-4	0	0	
c_B	Basic Variables B	W_1^1	W_1^2	W_1^3	W_2^1	W_2^2	W_2^3	W_2^4	W_2^5	S_1	S_2	Solution Values b=(X_B)
2	W_2^2	-32	-30	0	0	1	2	3	4	1	0	0
0	S_2	62	59	0	0	0	4	12	24	-2	1	30
6	W_1^3	1	1	1	0	0	0	0	0	0	0	1
0	W_2^1	32	30	0	1	0	-1	-2	-3	-1	0	1
z_j-c_j		-58	-58	0	0	0	2	6	12	2	0	

W_2^1 departs and W_1^2 enters.

Table 4

	c_j	0	4	6	0	2	2	0	-4	0	0	
c_B	Basic Variables B	W_1^1	W_1^2	W_1^3	W_2^1	W_2^2	W_2^3	W_2^4	W_2^5	S_1	S_2	Solution Values b=(X_B)
2	W_2^2	0	0	0	1	1	1	1	1	0	0	1
0	S_2	-14/15	0	0	-59/30	0	179/30	239/15	299/10	-1/30	1	841/30
6	W_1^3	-1/15	0	1	-1/30	0	1/30	1/15	1/10	1/30	0	29/30
4	W_1^2	16/15	1	0	1/30	0	-1/30	-1/15	-1/10	-1/30	0	1/30
z_j-c_j		58/15	0	0	29/15	0	1/15	32/15	31/5	1/15	0	

Thus, the optimum values of the weights are:
$W_1^2 = 1/30$, $W_1^3 = 29/30$ & $W_2^2 = 1$

The optimum values of the decision variables are:

$x_1 = \sum_{k=1}^{3} a_1^k W_1^k = a_1^3 W_1^3 = 2 X 29/30 = 1.9333 = 1.93$ (approx.)

$x_2 = \sum_{k=1}^{5} a_2^k W_2^k = a_2^2 W_2^2 = 1 X 1 = 1$

The maximum value of the objective function is
$5 X 1.93 - (1.93)^2 + 3 X 1 - (1)^2 = 7.9251$

Self Test Questions

Theory

1. Write short notes on the following:

- Nonlinear programming
- Quadratic programming
- Separable programming

2. Give the mathematical formulation of a general nonlinear programming problem.

Practical

Solve by quadratic programming method

1. Maximize $f(x) = 2x_1 + 4x_2 - x_1^2 - x_2^2$

Subject to
$x_1 + 4x_2 \leq 5$
$2x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0.$

2. Maximize $f(x) = x_1 + x_2 + x_3 - 1/2(x_1^2 + x_2^2 + x_3^2)$

Subject to
$x_1 + x_2 + x_3 \leq 1$
$4x_1 + 2x_2 \leq 7/3$

$x_1, x_2, x_3 \geq 0.$

3. Maximize $2x_1x_2 + 4x_2 - x_2^2 - 2x_1^2$

Subject to
$x_1 + 4x_2 \leq 12$

$x_1, x_2 \geq 0.$

4. Maximize $6x_1 + 4x_2 + 2x_3 - 3x_1^2 - 2x_2^2 - 1/3x_3^2$

Subject to
$x_1 + 2x_2 + x_3 \leq 4$

$x_1, x_2, x_3 \geq 0.$

Solve the following nonlinear programming problem by separable programming method:

1. Maximize $f_0 = 2x_1 - x_1^2 + x_2$

Subject to
$f_1 = 2x_1^2 + 3x_2^2 \leq 6$
$f_2 = x_1 \leq 2$
$f_3 = x_2 \leq 2$

$x_1, x_2 \geq 0.$

Take the break points of both x_1 and x_2 as 0, 1, 2, 3, 4.

2. Minimize $x_1^2 - 4x_1 + x_2^2 - 2x_3$

Subject to
$x_1 + x_2 + x_3 \leq 2$
$(x_1 + 1)\, x_2 \geq 2$

$x_1, x_2, x_3 \geq 0.$

Take breakpoints of x_1 and x_2 as 0, 1, 2. Keep x3 unchanged.

Chapter – 16

Simulation Models

Introduction

Simulation is a technique which describes a process by developing a model of that process, and then performing experiments on the model to predict the behaviour of the process over time.

An example of simulation is in computer games (e.g., chess, field combat war games, etc.). If the sequence of events in such games were predetermined, the player would quickly learn the sequence and become bored. One solution would be to have a large number of games stored in the program, but this could take up an inordinate amount of memory space. The usual solution is for the game program to choose its own moves at random. In most games, the total number of possible combinations of events or moves is so astronomically large that this method results in each game being unique. Some other examples of a simulated environment are planetarium shows and the environments in a museum.

Definition

Simulation starts when all else fails, i.e., it is a "**Method of Last Resort**".

Simulation is a technique of problem solving based upon experimentation performed on a model of real world situation.

Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behavior and structure of a complex, real world system over extended period of time.

When simulation should be used?

1. Actual observation of a system may be too expensive.
2. The problem is too big or intricate to handle with linear, dynamic, and standard probabilistic models.
3. The standard sensitivity analysis is too clumsy and computationally burdensome for observing the actual environment.

4. It is not possible to develop a mathematical model. Even though a mathematical model can be formulated, a straight forward analytical solution may not be available.
5. It is not possible to perform validating experiments on mathematical models describing the system.
6. There may not be sufficient time to allow the system to operate extensively.

Steps In The Simulation Process

Steps

1. Formulate the model. This step is almost the same as that for other operations research models. For a proper formulation, comprehensive study should be made regarding components of the problem, objective, composition of the organisation, etc.

2. Design the experiment. After building a simulation model, simulation experiments must be designed. In this step, you must decide the starting conditions of the model, parameter settings, time period required for each run, total number of runs, etc.

3. Develop the Computer Program: Using a high-speed electronic computer, simulation experiments can be performed. If the simulated model has a very simple structure, you can use a standard programming language, such as FORTRAN, PL/1, or ALGOL, to develop the computerized version. On the other hand, for a complex structure, you can use simulation languages like SIMSCRIPT or GPSS.

Monte-Carlo Simulation

The Monte-Carlo simulation method uses random numbers for generating some data by which a problem can be solved. These random numbers are helpful in creating a new set of hypothetical data for a problem whose behavior is known from past experience. The random numbers are generated either on a computer or are picked up from a table. Most computers employ what is known as pseudorandomness. This means that the numbers are generated by a series of specific operations. Each number is generated by performing these operations on the previous number. After picking a random number, its value is compared with the cumulative probability distribution and the value of process parameters is obtained.

The Lajwaab Bakery Shop Problem

Example

The Lajwaab Bakery Shop keeps stock of a popular brand of cake. Previous experience indicates the daily demand as given below:

Daily demand	Probability
0	0.01
15	0.15
25	0.20
35	0.50
45	0.12
50	0.02

Consider the following sequence of random numbers:
21, 27, 47, 54, 60, 39, 43, 91, 25, 20

Using this sequence, simulate the demand for the next 10 days. Find out the stock situation, if the owner of the bakery shop decides to make 30 cakes every day. Also estimate the daily average demand for the cakes on the basis of simulated data.

Solution:

Using the daily demand distribution, we obtain a probability distribution as shown in the following table.

Table 1

Daily demand	Probability	Cumulative probability	Random Numbers
0	0.01	0.01	0
15	0.15	0.16	1-15
25	0.20	0.36	16-35
35	0.50	0.86	36-85
45	0.12	0.98	86-97
50	0.02	1.00	98-99

At the start of simulation, the first random number 21 generates a demand of 25 cakes as shown in table 2. The demand is determined from the cumulative probability values in table 1. At the end of first day, the closing quantity is 5 (30-25) cakes.

Similarly, we can calculate the next demand for others.

Table 2

Demand	Random Numbers	Next demand	Daily production = 30 cakes	
			Left out	Shortage
1	21	25	5	
2	27	25	10	
3	47	35	5	
4	54	35	0	
5	60	35		5
6	39	35		10
7	43	35		15
8	91	45		30
9	25	25		25
10	20	25		20
Total		**320**		**10**

Total demand = 320
Average demand = Total demand/no. of days
The daily average demand for the cakes = 320/10 = 32 cakes.

Simulation and Inventory Control

Example

Zicom Electronics wants to determine the order size for calculators. The demand and lead time are probabilistic and their distributions are given below:

Demand / week (thousands)	Probability	Lead time	Probability
0	0.2	2	0.3
1	0.4	3	0.4
2	0.3	4	0.3
3	0.1		

The cost of placing an order is Rs. 100 per order and the holding cost for 1000 calculators is Rs. 2 per week. The shortage cost is Rs. 20 per thousand. Whenever the inventory level is equal to or below 2000, an order is placed equal to the difference between the current inventory balance and specified maximum replenishment level equal to 4000.

Simulate the policy for 10 weeks. Assume the following

- the beginning inventory is 3000 units
- no back orders are permitted
- each order is placed at the beginning of the week following the drop in inventory level to (or below) the reorder point
- the replenishment orders are received at the beginning of the week.

Solution:

Using the daily demand and lead time distributions, we assign a set of random numbers to represent the range of values of variables as shown in table 1 & table 2.

Table 1

Demand / week (thousands)	Probability	Cumulative Probability	Random Numbers
0	0.2	0.2	00-19
1	0.4	0.6	20-59
2	0.3	0.9	60-89
3	0.1	1.0	90-99

Table 2

Lead time (weeks)	Probability	Cumulative Probability	Random Numbers
2	0.3	0.3	00-29
3	0.4	0.7	30-69
4	0.3	1.0	70-99

At the start of simulation, the first random number 31 generates a demand of 1000 units, as shown in **table 3**. The demand is determined from the cumulative probability values in table 1. At the end of first week, the closing balance is 2000 units, which is equal to the reorder level; therefore, an order for 2000 (4000-2000) units is placed. The random number generated is 29, so the lead time is 2 weeks. The lead time is determined from the cumulative probability values in table 2. Since closing balance is 2000 units, the holding cost is Rs. 4

In the second week, the random number 70 generates a demand of 2000 units. Therefore, the closing balance at the end of second week is reduced to zero units.

In the third week, the demand for 1000 units can't be fulfilled because the available inventory is zero. This results in the shortage cost of Rs. 20.

The 2000 units ordered in the first week are received at the beginning of fourth week. The random number 86 generates a demand of 2000 units, and, hence closing stock is zero. Therefore, an order for 4000 (4000-0) units is placed. The random number generated is 83, so the lead time is 4 weeks. Therefore, the second shortage occurs in the fifth week. The units ordered at the end of fourth week are received in the beginning of ninth week.

Table 3

Week	Opening Balance Inventory ('000)	Demand		Closing Balance Inventory ('000)	Lead Time		Quantity Ordered ('000)	Costs	
		Random Numbers	Units ('000)		Random Numbers	Weeks		Holding Cost (Rs.)	Shortage Cost (Rs.)
1	3	31	1	2	29	2	2	4	-
2	2	70	2	0	-	-	-	-	-
3	0	53	1	-1	-	-	-	-	20
4	2*	86	2	0	83	4	4	-	-
5	0	32	1	-1	-	-	-	-	20
6	0	78	2	-2	-	-	-	-	40

7	0	26	1	-1	-	-	-	-	20
8	0	64	2	-2	-	-	-	-	40
9	4*	45	1	3	-	-	-	6	-
10	3	12	0	3	-	-	-	6	-
Total			**13**			**6**			

Note: * includes order quantity just received.

Average Inventory = 8000/10 = 800 units.
The average inventory is calculated by adding the closing inventory balances (ignoring negative balances) and dividing by the number of weeks.

Weekly average cost = Ordering Cost + Inventory Holding Cost + Shortage Cost

Ordering Cost = (100 X 2)/10
= Rs. 20

Inventory Holding Cost = (800 X 2)/1000
= Rs. 1.60

Shortage Cost = [20 X (1 + 1 + 2 + 1 + 2)]/10 = Rs. 14

Weekly average cost = Rs. 20 + Rs. 1.60 + Rs. 14
= Rs. 35.60

It should be noted that the shortage cost is high as compared to holding cost. The shortage cost can be reduced by increasing the reorder level.

Average lead time = 6/2 = 3 weeks
Average demand per week = 13000/10 = 1300 units
Average demand during lead time = 3 X 1300 = 3900 units
Maximum lead time = 4 weeks
Maximum weekly demand = 2000 units
Maximum demand during lead time = 4 x 2000 = 8000 units

Thus, the best reorder point should be somewhere between 3900 to 8000 units.

Simulation And Queuing System

Example 1

People arrive at the New Delhi Railway station to buy tickets according to the following distribution.

Inter-arrival Time (Min.)	Frequency
2	10
3	20
4	40
5	20
6	10

The service time is 5 minutes and there is only one ticket counter. The Railway station incharge is interested in predicting the operating characteristics of this counter during a typical operating day from 10.00 a.m. to 11.00 a.m. Use simulation to determine the average waiting time before service and average time a person spends in the system.

Solution:

From the given distribution of arrivals, the random numbers can be assigned to the arrival times as shown in **table 1**.

Table 1

Inter-arrival Time (Min.)	Frequency	Probability	Cumulative Probability	R.No.
2	10	0.10	0.10	0 - 09
3	20	0.20	0.30	10 - 29
4	40	0.40	0.70	30 - 69
5	20	0.20	0.90	70 - 89
6	10	0.10	1.00	90 - 99

The first random number generated is 17, which corresponds to the inter-arrival time of 3 minutes. This implies that the first person arrives 3 minutes after the service window opens, as shown in **table 2**. Since the first person arrives at 10.03 a.m., therefore, the server has to wait for 3 minutes. The server takes 5 minutes and thus, the first person leaves the system at 10.08 a.m. (10.03 + .05). Similarly, other values can be calculated.

Table 2

S.No.	R.No.	Inter-Arrival Time	Arrival Time	Service Starts	Service Ends	Waiting Time	
						Server	Person
1	17	3	10.03	10.03	10.08	3	-
2	86	5	10.08	10.08	10.13	-	-
3	84	5	10.13	10.13	10.18	-	-
4	79	5	10.18	10.18	10.23	-	-
5	33	4	10.22	10.23	10.28	-	1
6	55	4	10.26	10.28	10.33	-	2
7	6	2	10.28	10.33	10.38	-	5
8	42	4	10.32	10.38	10.43	-	6
9	93	6	10.38	10.43	10.48	-	5
10	38	4	10.42	10.48	10.53	-	6
11	58	4	10.46	10.53	10.58	-	7
12	71	5	10.51	10.58	11.03	-	7
Total							**39**

Average waiting time before service.
= Total waiting time (person)/Total no. of arrivals
= 39/12 = 3.25 minutes.

Average time a person spends in the system.
= Service time + Average waiting time before service
= 5 + 3.25 = 8.25 minutes.

Example 2

Students arrive at the head office of www.universalteacher.com according to the following arrival and service time probability distributions:

Inter-arrival Time (Min.)	Frequency	Service Time (Min.)	Frequency
1	1	1	1
2	4	2	4
3	7	3	7
4	17	4	17
5	31	5	31
6	23	6	23
7	7	7	7
8	5	8	5
9	3	9	3
10	2	10	2

Using Monte Carlo Simulation, determine the following:
(a) The average waiting time before service
(b) The average time a student spends in the system

Solution.

From the given distribution of arrivals, the random numbers can be assigned to the arrival times as shown in **table 1**.

Table 1

Inter-arrival Time (Min.)	Frequency	Probability	Cumulative Probability	R.No.
1	1	0.01	0.01	0
2	4	0.04	0.05	1 - 4
3	7	0.07	0.12	5 - 11
4	17	0.17	0.29	12 - 28
5	31	0.31	0.60	29 - 59
6	23	0.23	0.83	60 - 82
7	7	0.07	0.90	83 - 89
8	5	0.05	0.95	90 - 94
9	3	0.03	0.98	95 - 97
10	2	0.02	1.00	97 - 99

Similarly, the random numbers can be assigned to the service times as shown in **table 2**.

Table 2

Service Time (Min.)	Frequency	Probability	Cumulative Probability	R.No.
1	1	0.01	0.01	0
2	4	0.04	0.05	1 - 4
3	7	0.07	0.12	5 - 11
4	17	0.17	0.29	12 - 28
5	31	0.31	0.60	29 - 59
6	23	0.23	0.83	60 - 82
7	7	0.07	0.90	83 - 89
8	5	0.05	0.95	90 - 94
9	3	0.03	0.98	95 - 97
10	2	0.02	1.00	97 - 99

The simulation worksheet as shown in table 3 is developed in the following way:

The first random number for inter-arrival time is 17, which corresponds to an inter-arrival time of 4 minutes. This implies that the first student arrives 4 minutes after the head office opens. Since the first student arrives at 10.04 a.m., therefore, the server has to wait for 4 minutes. The random number for service time is 90, which corresponds to a service time of 8 minutes. Thus, the first student leaves the system at 10.12 a.m. (10.04 + .08).

The next random number for inter-arrival time is 86, which corresponds to an inter-arrival time of 7 minutes. Since the second student arrives at 10.11 a.m., but the service can be started at 10.12 a.m. Therefore, he has to wait for 1 minute. Likewise, other values can be calculated.

Table 3

S.No.	R.No.	Inter-arrival time	Arrival Time	Service starts	R.No.	Service		Waiting Time	
						Time	Ends	Server	Student
1	17	4	10.04	10.04	90	8	10.12	4	-
2	86	7	10.11	10.12	59	5	10.17	-	1
3	84	7	10.18	10.18	95	9	10.27	1	-
4	79	6	10.24	10.27	68	6	10.33	-	3
5	33	5	10.29	10.33	51	5	10.38	-	4
6	55	5	10.34	10.38	82	6	10.44	-	4
7	6	3	10.37	10.44	72	6	10.50	-	7
8	42	5	10.42	10.50	1	2	10.52	-	8
9	93	8	10.50	10.52	77	6	10.58	-	2
10	38	5	10.55	10.58	80	6	11.04	-	3
Total						**59**			**32**

Average waiting time before service.
= 32/10 = 3.2 minutes.

Average service time.
= 59/10 = 5.9 minutes

Average time a student spends in the system.
5.9 + 3.2 = 9.1 minutes.

Simulation And Capital Budgeting

Example

The Yum Yum corporation is considering the problem of marketing a new chocolate. The investment required in the project is Rs. 2,00,000. There are two factors that are uncertain - annual demand & profit. The management has the past data regarding the possible levels of two factors.

Annual Demand	Probability	Profit	Probability
1000	0.10	3.00	0.10
2000	0.20	5.00	0.20
3000	0.40	7.00	0.40
4000	0.20	9.00	0.20
5000	0.10	10.00	0.10

Using Monte-Carlo Simulation, determine the following:

- return on investment
- average profit
- average demand

Solution.

Table

S.No.	Random No.	Simulated Demand (SD)	Random No.	Simulated Profit (SP)	Return (%) (SD X SP X 100) / 200000
1	35	3000	15	5	7.5
2	55	3000	80	9	13.5
3	10	2000	50	7	7.0
4	30	3000	90	10	15
5	70	4000	30	7	14
6	90	5000	60	7	17.5
7	25	2000	25	5	5
8	52	3000	62	7	10.5
9	62	3000	10	5	7.5
10	31	3000	2	3	4.5
Total		31000		65	

Average profit = 65/10 = 6.5.
Average demand = 31000/10 = 3100.

Limitations of Simulation

- A good simulation model may be very expensive. Simulation often requires a significant amount of computer time and is therefore expensive.
- Simulation generates a way of evaluating solutions but does not generate solutions themselves.
- Simulation is not precise. It does not yield an answer but merely provides a set of the system's responses to different operating conditions. In many cases, this lack of precision is difficult to measure.
- It is a trial and error method that may produce different solutions in repeated runs.
- The difficulty in finding the optimal values increases due to an increase in the number of parameters.

Self Test Questions

Theory

1. Define simulation.

2. What are the advantages and limitation of simulation?

Practical

1. At a service station, a study was made over a period of 25 days to determine both the number of automobiles being brought in service and the number of automobiles serviced. The results are given below:

No. of automobiles arriving for service or completing service per day	Frequency of arrivals for service	Frequency of daily completion
0	2	3
1	4	2
2	10	12
3	5	3
4	3	4
5	1	1

Simulate the arrival pattern for a ten day period and estimate the mean number of automobiles that remain in service for more than one day.

2. The Lajwaab Bakery House keeps stock of a popular brand of cake. Daily demand based on past experience indicates

Daily Demand	:	0	15	25	35	45	50
Probability	:	.01	.15	.20	.50	.12	.02

Consider the following sequence of random numbers:-
R. No. 48, 68, 09, 51, 56, 70, 15, 34, 68, 19, 22, 90, 30, 41, 50
Using this sequence, simulate the demand for the next 15 days

Find out the stock situation, if the owner of the bakery decides to buy 30 cakes every day. Also estimate the daily average demand for the cakes on the basis of simulated data.

3. The occurrence of rain in Delhi on a day is dependent upon whether or not it rained on the previous day. If it rained on the previous day, the rain distribution is given by:

Event	Probability
No. rain	0.55
1 cm. rain	0.20
2 cm. rain	0.15
3 cm. rain	0.05
4 cm. rain	0.03
5 cm. rain	0.02

If there was no rain on the previous day, the rain distribution is given by:

Event	Probability
No. rain	0.70
1 cm. rain	0.20
2 cm. rain	0.06
3 cm. rain	0.04

Simulate Delhi's weather for 10 days and determine by simulation the total days without rain as well as the total rainfall during the period.

Use the following random numbers:
48, 68, 09, 51, 56, 90, 15, 34, 68, 19

Assume that for the first day of the simulation it had not rained the day before.

4. Dr. Dang is a dentist who schedules all his patients for 30 minutes appointments. Some of the patients take more or less than 30 minutes depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and the time needed to complete the work:

Category	Time required	Probability of categories
Filling	45 minutes	0.40
Cleaning	15 minutes	0.25
Extraction	45 minutes	0.15
Checkup	15 minutes	0.20

Simulate the dentist's clinic for five hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that the clinic opens at 8.00 a.m.

Use the following random numbers for handling the above problem:
40, 82, 11, 34, 25, 66, 17, 79, 48, 68, 09, 51, 56, 90, 15

5. The management of Spitzen Watch company is considering the problem of marketing a new product . The fixed cost required in the project is Rs. 3,000. Three factors are uncertain viz. the selling price, variable cost and the annual sales volume. The product has a life of only one year. The management has the data on these three factors as under:

Selling Price Rs.	Probability	Variability	Probability	Sales Volume (Units)	Probability
3	0.2	1	0.3	2,000	0.3
4	0.5	2	0.6	3,000	0.3
5	0.3	3	0.1	5,000	0.4

Consider the following sequence of random numbers:
81, 32, 60, 04, 46, 31, 67, 25, 24, 10

Using the above sequence, simulate the average profit for the above project on the basis of 10 trials.

THE END

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